Department of Mathematics & Statistics

Ph.D admission written test

Time: 1 Hour **July 1, 2016**

Total Marks: 36

NAME:____

Instructions

1. Write your name in **CAPITAL** letters.
2. We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the set of natural numbers, integers, rational numbers, real numbers and complex numbers respectively.
3. Each question carries 3 marks. **No** negative marks.
4. There are two multiple choice questions, six fill in the blanks questions, two true or false questions and two computational questions.
5. These question-cum-answer-sheets must be returned to the invigilator before leaving the examination hall.

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1. If $z = z(x, y)$ satisfies
$$F\left(\frac{x}{y}, \frac{z}{y}\right) = 0,$$
where F is an arbitrary differentiable function, then z satisfies the first order partial differential equation __________.

   Ans. $xz_x + yz_y = z/px + qy = z$

2. Consider the data given below:

x	-2	-1	0	1	3
y	17	9	3	-1	-3

   The degree of the polynomial which interpolates the data is __________.

   Ans. 2

3. Let $y = \sin(x) + xe^x$ be a solution of the fourth order ordinary differential equation $a_4\, y^{(4)} + a_3\, y^{(3)} + a_2\, y^{(2)} + a_1\, y^{(1)} + a_0\, y = 0$, where $a_i,\ i = 0, 1, \cdots, 4$ are constants. Then $a_4 + a_2 =$ __________.

   Ans. 3

4. True or False. Let (a_1, a_2, a_3, a_4) be a non-zero vector in $\mathbb{R}^4$. There exists a one-one linear map $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0\}$ is the image of T.

   Ans: False

5. True or False. If A and B are two $n \times n$ matrices with same characteristic and same minimal polynomial then they have the same Jordan form.

   Ans: False

6. Let $f : \mathbb{Q} \to \mathbb{Z}$ be a homomorphism of additive groups. Then $f(r) =$ _____________ for all $r \in \mathbb{Q}$.

7. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a differentiable function such that $| f(x) | \leq || x ||^{2016}$ for $x \in \mathbb{R}^2$. Then $(\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)) =$ _____________.

   Ans. Here $f(0,0) = 0$. Then from the limit definition it follows that $f'(0,0)$ is the zero map $\mathbb{R}^2 \longrightarrow \mathbb{R}$.

8. Compute the line integral $\int_\gamma x dz$ where γ is the line segment from 0 to $1 + i$.

   Ans: (1+i)/2.

9. Suppose $f : \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\} \to \mathbb{D}$ is analytic map such that $f(0) = 0$ and $f(\frac{1}{2}) = \frac{1}{2\sqrt{2}} + i\frac{1}{2\sqrt{2}}$. Find $f(1/4)$.

10. Let A and B be non-empty subsets of real numbers. Which of the following statement(s) is(are) equivalent to saying that LUB(A) $\leq$ LUB(B)?

    (a) For every $a \in A$ and $\epsilon > 0$ there exists a $b \in B$ such that $a < b + \epsilon$.

    (b) For every $b \in B$ there exists an $a \in A$ such that $a \leq b$.

    (c) There exists $b \in B$ such that $a \leq b$ for all $a \in A$.

    (d) There exists $a \in A$ such that $a \leq b$ for all $b \in B$. Ans: (a).

11. Let τ_1 be the topology on $\mathbb{R}$ generated by the base $\mathcal{B} = \{[a,b) : a < b \in \mathbb{R}\}$. If τ_0 is the standard topology on $\mathbb{R}$ and $\mathrm{Id} : \mathbb{R} \to \mathbb{R}$ is the identity mapping then

    (a) $\mathrm{Id} : (\mathbb{R}, \tau_1) \to (\mathbb{R}, \tau_0)$ is continuous but not an open mapping.

    (b) $\mathrm{Id} : (\mathbb{R}, \tau_1) \to (\mathbb{R}, \tau_0)$ is an open mapping but not continuous.

    (c) $\mathrm{Id} : (\mathbb{R}, \tau_1) \to (\mathbb{R}, \tau_0)$ is a homeomorphism.

    (d) $\mathrm{Id} : (\mathbb{R}, \tau_1) \to (\mathbb{R}, \tau_0)$ is neither continuous nor an open mapping. Ans: (a)

12. Let $\mathbb{F}$ be the set of irrational numbers in $\mathbb{R}$ and $X := \mathbb{R}^2 \setminus (\mathbb{F} \times \mathbb{F})$ with usual subspace topology of $\mathbb{R}^2$. The number of connected components of X is _____________.