Hyperbolicity properties of extensions of free groups

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Consider a free group of finite rank $\mathbb{F}$ and the natural short exact sequence $1 \to \text{Inn}(\mathbb{F}) \to \text{Aut}(\mathbb{F}) \to \text{Out}(\mathbb{F}) \to 1$. Given a finitely generated subgroup $Q < \text{Out}(\mathbb{F})$, we get an induced short exact sequence $1 \to \mathbb{F} \to \Gamma_Q \to Q \to 1$ by identifying $\mathbb{F}$ with $\text{Inn}(\mathbb{F})$. We are interested in understanding what conditions can we put on Q so that Γ_Q (called the extension of $\mathbb{F}$ by Q) has a nice geometric structure (specifically - hyperbolic or relatively hyperbolic group structure).

1. When $Q \cong \mathbb{Z}$: Γ_Q is hyperbolic $\Leftrightarrow$ Γ_Q does not have a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup (Bestvina-Feighn, Brinkmann) and Γ_Q is relatively hyperbolic $\Leftrightarrow$ the generator of Q is *exponentially growing* element of $\text{Out}(\mathbb{F})$ (— , Hagen).
2. When $\langle \phi \rangle \cong \mathbb{Z} \cong \langle \psi \rangle$ are *atoroidal* with their lamination sets disjoint, then sufficiently high powers of ϕ, ψ generate a free group of rank two Q, so that Γ_Q is hyperbolic (— - Gultepe). Converse is also holds (—).
3. When $\langle \phi \rangle \cong \mathbb{Z} \cong \langle \psi \rangle$ are *exponentially growing*, a slightly technical notion of *relative independence* ensures that we have a free group of rank two Q generated by sufficiently high powers of ϕ, ψ, so that Γ_Q is a relatively hyperbolic group. Converse also holds (— - Gultepe).

We will try to keep the technicalities to a bare minimum and understand the key ideas through examples and pictures.