Major contributions of Pravir Dutt, Arbind Lal and Sudipta Dutta

  

Prepared by
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together with their collaborators

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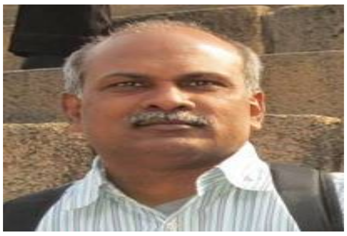


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Preface

In the recent past, we have lost three very dear colleagues, Prof. Pravir Dutt, Prof. Arbind Lal and Prof. Sudipta Dutta tragically. All three, apart from being wonderful people, were also brilliant mathematicians. Each of them has left a lasting impression on their students, colleagues and collaborators. They have indeed inspired, motivated and trained their students so well, that many of them are competent teachers and mathematicians today.

While, these losses still appear very unreal, we have decided to celebrate their mathematical legacy, by organizing this symposium in their memory and honor. We are especially thankful to Prof. Pradipta Bandhopadhyay, Prof. Sukanta Pati and Prof. Akash Anand for preparing this booklet. They have interacted with various collaborators of Prof. Dutt, Prof. Lal and Prof. Dutta to make this booklet. We express our sincere gratitude to each of them for all their efforts.

Chapter 1

A glimpse of Sudipta Dutta's work:

Contributions to geometry of Banach spaces

Abstract *This is a survey of the contribution of Sudipta Dutta. Sudipta and his coauthors have contributed to several aspects of Geometry of Banach spaces, both to the structural and technical aspects. Some of these include, aspects of Approximation theory, almost constrained (AC) subspaces, geometry of the dual unit ball, understanding L_1-predual spaces via representing matrices, algebraic reflexivity of sets of operators on classical function spaces, G_δ-embeddings and structure theory of L^p-spaces.*

Professor Sudipta Dutta was born on August 29, 1976. He did his B. Stat. (1997), M. Stat. (1999) and Ph. D. (2004) from Indian Statistical Institute (ISI). Then he joined Ben Gurion University of the Negev, Israel, as a post-doctoral fellow during 2004-06. During this period, he also visited University of Paris VI. Returning briefly to ISI in 2006, he joined IIT Kanpur as Assistant Professor in July, 2007, and became a Professor in 2016.

Two students got their Ph. D. degrees under his supervision. He was supervising a third student at the time of his demise.

Sudipta's research interests included Banach space theory, Operator spaces and Abstract harmonic analysis. He was quite well-known in his area. In the short span of less than 15 years, he published about 29 articles (4 of them posthumous), all in premier journals. According to MathSciNet, his works have been cited 137 times by 95 authors. He was awarded the Indo-U.S. Science and Technology Fellowship in 2008-09, and the P K Kelkar Young Research Fellowship of IIT Kanpur.

Starting from his pre-Ph. D. days till the very end, he had collaborated with others on problems in somewhat disjoint areas, parallel to the works that went into his or his students' Ph. D. thesis. The list of his collaborators are thus quite impressive. So is the breadth of the areas in which he has contributed as can be seen from the AMS subject classifications of his papers.

We list his papers chronologically in the References, but rearrange them thematically in this write-up.

Let us begin with the papers that went into his Ph. D. thesis, titled "Intersection Properties of Balls in Banach Spaces and Related Topics", or at least had their origin in the thesis.

To fix our notations, for a Banach space X, $B(X) = \{x \in X : \|x\| \leq 1\}$ and $S(X) = \{x \in X : \|x\| = 1\}$. For $x \in S(X)$, $D(x) = \{f \in S(X^*) : f(x) = 1\}$, the duality map. And for $f \in S(X^*)$, $D^{-1}(f) = \{x \in S(X) : f(x) = 1\}$, the pre-duality map. Note that $D^{-1}(f)$ may be empty.

[4] Pradipta Bandyopadhyay, S. Dutta, *Almost constrained subspaces of Banach spaces*, Proc. Amer. Math. Soc. 132 (2004), no. 1, 107–115.

[14] Pradipta Bandyopadhyay, S. Dutta, *Almost constrained subspaces of Banach spaces—II*, Houston J. Math. 35 (2009), no. 3, 945–957.

A subspace Y of a Banach space X is almost constrained (AC) if any family of closed balls centred at points of Y that intersects in X also intersects in Y. The paper [4] discusses AC subspaces of Banach spaces, gives an example to show that an AC-subspace need not, in general, be 1-complemented, and obtains sufficient conditions for an AC-subspace to be 1-complemented. This condition is in terms of functionals with "locally unique" Hahn-Banach extensions.

The paper [14] continues this study. In this paper, the authors show that a subspace H of finite codimension in $C_{\mathbb{C}}(K)$ is an AC-subspace if and only if H is 1-complemented. They also give a simple proof that the implication "$AC \implies$ 1-complemented" holds for any subspace of $c_0(\Gamma)$ and c. Only a part of this paper went into Sudipta's thesis.

[2] P. Bandyopadhyay, S. Basu, S. Dutta, B.-L. Lin, *Very non-constrained subspaces of Banach spaces*, Extracta Math. 18 (2003), no. 2, 161–185.

It is well known that every dual Banach space X^* is 1-complemented in X^{***}. In sharp contrast, some Banach spaces are "nicely smooth", i.e., given

$F, G \in X^{**}$, there is a ball centred at a point of X which contains F but not G. This paper investigates the more general situation of a "very non-constrained" (VN) subspace Y of a Banach space X, and shows that most results extend through similar techniques. A fundamental feature of this paper is the algebraic nature of the treatment which considerably simplifies the proofs. One of the main results states, *inter alia*, that Y is VN in X if and only if no nonzero vector in X is Birkhoff-orthogonal to Y. It is shown that a hyperplane is either a VN-subspace or an AC-subspace and in the later case, it is constrained. Some stability results are also proved.

[1] Pradipta Bandyopadhyay, S. Dutta, *Weighted Chebyshev centres and intersection properties of balls in Banach spaces*, Function spaces (Edwardsville, IL, 2002), 43–58, Contemp. Math., 328, Amer. Math. Soc., Providence, RI, 2003.

This paper studies weighted Chebyshev centres and their relationship with intersection properties of balls. Veselý [48] has studied Banach spaces that admit weighted Chebyshev centres for finite sets. Subsequently, Bandyopadhyay and Rao [30] had shown, *inter alia*, that L_1-preduals have this property. Extending these results, this paper explores when a general family of sets admit weighted Chebyshev centres, the typical examples being families of finite, compact, bounded or arbitrary sets. A new feature of this treatment is relating this question with the notion of minimal points. An interesting result here is if X^{**} is strictly convex and X admits weighted Chebyshev centres for all compact sets, then for $A \subseteq X$ compact, the set of minimal points of A is weakly compact. This strengthens a result of Beauzamy and Maurey [31] with a much simpler proof.

[6] Pradipta Bandyopadhyay, S. Dutta, *Farthest points and the farthest distance map*, Bull. Austral. Math. Soc. 71 (2005), no. 3, 425–433.

Interesting results of this paper include the observation that if X is strictly convex, then every (weakly) compact convex set in X is the closed convex hull of its farthest points if and only if every such set is the intersection of closed balls containing it; and if X has the Radon-Nikodým Property (RNP), then similar result holds for w*-compact convex sets in X^*. A notion of strongly farthest points is introduced and strictly (resp., locally uniformly) convex spaces are characterized as those for which every farthest point of a compact (resp., closed bounded) convex set is a strongly farthest point. The authors also obtain an expression for the subdifferential of the farthest distance map in the spirit of Preiss's theorem [45] showing that the subdifferential of the farthest distance map is the unique maximal-monotone extension of a densely defined monotone operator involving the duality map and the farthest point map.

[7] S. Dutta, *Generalized subdifferential of the distance function*, Proc. Amer. Math. Soc. 133 (2005), no. 10, 2949–2955.

In this paper, the *proximal normal formula* is derived for almost proximinal sets in a smooth and locally uniformly convex Banach space improving upon earlier results of Borwein and Giles [32]. A necessary and sufficient condition is obtained for the convexity of Chebyshev sets in a Banach space X with both X and X^* locally uniformly convex, weakening the reflexivity assumptions in similar situations.

One of his earliest work outside his thesis was with Prof. T. S. S. R. K. Rao of ISI, Bangalore.

[3] S. Dutta, T. S. S. R. K. Rao, *On weak*-extreme points in Banach spaces*, J. Convex Anal. 10 (2003), no. 2, 531–539.

This paper is concerned with extreme points of $B(X)$ that remain extreme in $B(X^{**})$ under the canonical embedding. Such points are called weak*-extreme. More generally, the authors study, among other things, for a subspace M of a given Banach space X, conditions under which extreme points of $B(M)$ belong to (or fail to belong to) the same class of extreme points in $B(X)$. For instance, it is shown that, if M is an M-ideal in X, then no weak*-extreme point of $B(M)$ can be weak*-extreme in $B(X)$. Many specific examples are discussed: There is a strictly convex space X such that every $x \in S(X)$ remain extreme in $B(X^{**})$ but are no longer extreme in $B(X^{(4)})$. Furthermore, conditions are given which ensure that a $T \in S(\mathcal{K}(X,Y))$ weak*-extreme or strongly extreme in $B(\mathcal{K}(X,Y))$.

The second section of the paper investigates the relationship between weak*-extreme points and very smooth points. In particular, under additional assumptions, those extreme points of $B(\mathcal{L}(X,Y)^*)$ are described which are points of weak*-weak continuity for the identity map. Finally, it is shown that, if every equivalent norm in a Banach space X has a very smooth point, then X^* has the RNP.

[5] S. Dutta, T. S. S. R. K. Rao, *Norm-to-weak upper semi-continuity of the pre-duality map*, J. Anal. **12** (2004), 115–124.

In this paper, the authors continued to explore the above kind of preserved 'extremal' behaviour by studying points of norm-weak upper semicontinuity of the pre-duality map D^{-1}. They used an alternate descriptions due to Godefroy and Indumathi [36] to show among other things, such points extend from the dual of an M-ideal to the whole space and if it is in a weak*-closed subspace, then it is a point of norm-weak usc w.r.t the quotient space.

[12] S. Dutta, T. S. S. R. K. Rao, *Algebraic reflexivity of some subsets of the isometry group*, Linear Algebra Appl. **429** (2008), no. 7, 1522–1527.

In this paper, the authors investigate the algebraic reflexivity of sets of operators. Let X be a complex Banach space. For $\mathcal{A} \subseteq \mathcal{L}(X)$ the algebraic closure of $\mathcal{A}$ is defined as follows:

$$\overline{\mathcal{A}}^a = \{T \in \mathcal{L}(X) : \forall\, x \in X, \exists\, A \in \mathcal{A} \text{ such that } A(x) = T(x)\}$$

We say that is $\mathcal{A}$ is algebraically reflexive if $\overline{\mathcal{A}}^a = \mathcal{A}$.

For a compact Hausdorff space K, the authors show that if the group of isometries of $C(K)$ is algebraically reflexive, then the set of involutary isometries is again algebraically reflexive. In case of $C(K, X)$, one additionally needs X to be uniformly convex.

The authors also studied generalized bi-circular projections (GBP) (see Definition [1.0.5] below) on the space $A(K, X)$ of vector-valued affine continuous functions on a Choquet simplex K with X^* strictly convex.

This paper is the second most cited paper by Sudipta with 13 citations.

One theme that appears repeatedly in Sudipta's work is proximinality. It began with two papers with D. Narayana and continued with four papers with P. Shunmugaraj and others.

A closed set $A \subseteq X$ is said to be proximinal if for every $x \in X$, $P_A(x) =: \{y \in A : \|x - y\| = d(x, A)\} \neq \emptyset$. A proximinal set A is said to be strongly proximinal if given $\varepsilon > 0$ there exists $\delta > 0$ such that $P_A(x, \delta) =: \{y \in A : \|x - y\| < d(x, A) + \delta\} \subseteq P_A(x) + \varepsilon B(X)$.

G. Godefroy and V. Indumathi [35] introduced strong proximinality and in a series of papers, have shown that certain geometric properties like strong subdifferentiability (SSD) and QP-points are related to the existence of strong proximinal subspaces of finite co-dimension.

[9] S. Dutta, D. Narayana, *Strongly proximinal subspaces of finite codimension in $C(K)$*, Colloq. Math. **109** (2007), no. 1, 119–128.

This paper continues the study of strongly proximinal subspaces. For finite codimensional subspaces of $C(K)$ spaces, it is shown that strong proximinality is a transitive property. An important step in establishing this is to prove that the metric projection onto any such subspace is continuous in the Hausdorff metric.

This paper is the most cited paper by Sudipta with 15 citations.

[10] S. Dutta, D. Narayana, *Strongly proximinal subspaces in Banach spaces*, Function spaces, 143–152, Contemp. Math., 435, Amer. Math. Soc., Providence, RI, 2007.

In this paper, the authors show that a proximinal subspace Y of finite co-dimension in $L_1(\mu)$-space is strongly proximinal if every hyperplane containing it is strongly proximinal.

It is also noted that the notion of local U-proximinality studied earlier by K. S. Lau [42] is equivalent to strong proximinality, thus answering a question raised by Godefroy and Indumathi.

This paper is also one of the five most cited papers by Sudipta.

[15] S. Dutta, P. Shunmugaraj, *Strong proximinality of closed convex sets*, J. Approx. Theory **163** (2011), no. 4, 547–553.

A sequence $(x_n) \subseteq M$ is said to be a minimizing sequence for $x \in X$ if $\|x - x_n\| \to d(x, M)$. The set M is said to be *approximatively (weakly) compact* if every minimizing sequence in M has a (weakly) convergent subsequence which converges (weakly) to an element of M.

The main result of this paper is

[1.0.1]. *The following statements are equivalent :*

(a) *The norm on X is strongly subdifferentiable at all points of $S(X)$ and $D^{-1}(f)$ is compact for every $f \in S(X^*)$.*

(b) *X is reflexive and the relative weak and norm topologies coincide in $S(X)$.*

(c) *Every closed convex subset of X is approximatively compact.*

(d) *Every closed convex subset of X is strongly proximinal.*

This paper is also one of the five most cited papers by Sudipta.

[17] S. Dutta, P. Shunmugaraj, *Modulus of strong proximinality and continuity of metric projection*, Set-Valued Var. Anal. **19** (2011), no. 2, 271–281.

In this paper, a quantitative study of strong proximinality is initiated. If M is a proximinal subspace of X, then the *modulus of strong proximinality* $\varepsilon : X \setminus M \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined by

$$\varepsilon(x,t) = \inf\{r > 0 : P_M(x,t) \subseteq P_M(x) + rB(M)\}.$$

It follows from the definition of the strong proximinality that a proximinal subspace of X is strongly proximinal if and only if for each $x \in X \setminus M$, $\varepsilon(x,t) \to 0$ as $t \to 0$. The authors also obtain the following

[1.0.2]. *Let M be a strongly proximinal subspace of X. Then P_M is continuous at $x \in X$ if and only if for every $t > 0$, $\varepsilon(\cdot, t)$ is continuous at x.*

Upper bounds of $\varepsilon(\cdot,\cdot)$ for proximinal subspaces of certain spaces such as spaces with $1\frac{1}{2}$-ball property, uniformly convex and $C(K)$ with finite co-dimension are also estimated.

[24] S. Dutta, P. Shunmugaraj, Vamsinadh Thota, *Uniform strong proximinality and continuity of metric projection*, J. Convex Anal. **24** (2017), no. 4, 1263–1279.

As mentioned before, it is noted in [10] that the notion of local U-proximinality defined by K. S. Lau [42] is equivalent to strong proximinality. In this paper, the authors consider the notion of U-proximinality considered in [42] and relate it to "uniform strong proximinality".

A proximinal subspace M of X is said to be *uniformly strongly proximinal* (USP) on a subset A of X if for $\varepsilon > 0$, there exists $\delta > 0$ such that $P_M(x,\delta) \subseteq P_M(x) + \varepsilon B(X)$ for every $x \in A$.

One of the main result of [24] is:

[1.0.3]. *Let M be a proximinal subspace of a Banach space X. Then the following statements are equivalent.*

(a) *M is U-proximinal in X.*

(b) *Given $\varepsilon > 0$ there exists $\delta > 0$ such that $(1+\delta)B(X) \cap [x+M] \subseteq (B(X) \cap [x+M]) + \varepsilon B(X)$ for all $x \in B(X)$.*

(c) *Given $\varepsilon > 0$ there exists $\delta > 0$ such that $(1+\delta)B(X) \cap [x+M] \subseteq (B(X) \cap [x+M]) + \varepsilon B(X)$ for all $x \in \{x \in S(X) : d(x,M) = 1\}$.*

(d) *M is uniformly strongly proximinal on $\{x \in S(X) : d(x,M) = 1\}$.*

As a consequence of Theorem [1.0.3], it is shown that U-proximinality is a natural sufficient condition for the (uniform) continuity of the metric projection. A characterization of uniformly convexity in terms of uniform strong proximinality is also established.

[25] S. Dutta, P. Shunmugaraj, *Weakly compactly LUR Banach spaces*, J. Math. Anal. Appl. **458** (2018), no. 2, 1203–1213.

This paper continues the study of relations between geometric properties of X and proximinality properties of $S(X)$.

A Banach space X is said to be *weakly CLUR* if whenever $x, x_n \in S(X)$ and $\|x_n + x\| \to 2$, (x_n) has a subsequence which converges weakly to an element of $A(x) = \cup\{D^{-1}(x^*) : x^* \in D(x)\}$.

The following result is established in this paper.

[1.0.4]. *Consider the following statements.*

(1) *X is weakly LUR.*

(2) *For every $x \neq 0$, $S(X)$ is approximatively weakly compact at x and $P_{S(X)}(x)$ is a singleton.*

(3) *X is weakly CLUR and strictly convex.*

(4) *X is weakly CLUR.*

(5) *For every $x \neq 0$, $S(X)$ is approximatively weakly compact at x.*

Then $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Rightarrow (4) \Leftrightarrow (5)$.

Several other geometric properties of weakly CLUR are also presented.

[8] S. Dutta, V. P. Fonf, *On Tauberian and co-Tauberian operators*, Extracta Math. 21 (2006), no. 1, 27–39.

A bounded linear operator $T \in L(X, Y)$ is called a Tauberian operator if $T^{**}(x^{**}) \in Y$ implies that $x^{**} \in X$. T is co-Tauberian if T^* is Tauberian. This work relates the existence of nontrivial Tauberian or co-Tauberian operators with structural properties of the Banach spaces involved. For instance, it is shown that a Banach space X contains an infinite-dimensional reflexive subspace if and only if there exists a one-to-one Tauberian operator with domain X which is not an isomorphism.

[11] S. Dutta, Alexandre Godard, *Banach spaces with property* (M) *and their Szlenk indices*, Mediterr. J. Math. 5 (2008), no. 2, 211–220.

In this paper, the authors considered the so-called property (M), which roughly means that from the point of view of asymptotic smoothness, all points of the sphere are equivalent. This property had been defined by Nigel Kalton and Dirk Werner [40] and it is quite frequently satisfied. This paper shows that Asplund spaces with Property (M) have a minimal Szlenk index, and moreover that a norm with property (M) has optimal asymptotic smoothness among all equivalent norms on given Banach space. Their results apply in particular to Orlicz spaces.

[13] S. Dutta, V. P. Fonf, *On tree characterizations of* G_δ*-embeddings and some Banach spaces*, Israel J. Math. 167 (2008), 27–48.

Let E, X be Banach spaces. A bounded linear one-to-one operator $T : E \to X$ is called a G_δ-embedding if for every norm closed bounded and separable subset $D \subseteq E$, $T(D)$ is a G_δ-subset of X.

A tree in a Banach space E is defined as a family (x_A) of elements of E indexed by finite subsets of $\mathbb{N}$. A sequence $\{x_{A_n}\}$ is called a branch of the tree if the cardinality of A_n is n and A_n is the initial segment of A_{n+1}. A tree (x_A) is called T-null if $\lim\limits_{n\to\infty} Tx_{A\cup\{n\}} = 0$ for all $A \subseteq \mathbb{N}$.

One of the main results of the paper is: Let E, X be separable Banach spaces and $T : E \to X$ be a one-to-one bounded linear operator. Then T is a G_δ-embedding if and only if every T-null tree in $S(E)$ has a branch which is a boundedly complete basic sequence.

The paper also contains some results on the point of continuity property and results showing that "tree-branch" assumptions in the results of this paper cannot be replaced by "sequence-subsequence" assumptions.

[20] S. Dutta, V. P. Fonf, *Boundaries for strong Schur spaces*, Q. J. Math. 65 (2014), no. 3, 887–891.

Let X be a real Banach space. A subset $B \subseteq S(X^*)$ is called a boundary for X if, for every $x \in X$ there is an $f \in B$ such that $f(x) = \|x\|$. It was proved by the second author in [34] that if X does not contain an isomorphic copy of c_0, then for any representation $B = \bigcup_{n=1}^{\infty} B_n$ of a boundary B for X such that the sequence B_n is increasing, there exist an index m and an $r > 0$ such that B_m is r-norming for X. In this paper, the authors show that if $r > 0$ can be chosen independently of a boundary B and its representation, then that property characterizes strong Schur spaces.

For $\delta > 0$, a Banach space is said to be δ-strong Schur, if, given $\varepsilon > 0$ and a sequence (x_n) such that $\|x_n - x_m\| > \varepsilon$, $n \neq m$, (x_n) has a subsequence which is $\varepsilon\delta$-equivalent to the unit vector basis of ℓ_1. We say that X is strong Schur if it is δ-strong Schur for some $\delta > 0$.

Let $0 < \alpha < 1$. We say that a Banach space X has the α-Schur property if every normalized sequence in X has a weak*-limit point $F \in X^{**}$ with $\|F\| \geq \alpha$.

The authors prove that the following are equivalent:

(a) X is a strong Schur space.

(b) There exists $r > 0$ such that, for any boundary B of X and representation $B = \bigcup_{n=1}^{\infty} B_n$ with B_n increasing, there is $m \in \mathbb{N}$ such that B_m is r-norming for X.

(c) X is α-Schur for some $\alpha > 0$.

Dr. Abdullah Bin Abu Baker was Sudipta's first Ph. D. student.

[16] A. B. Abubaker, S. Dutta, *Projections in the convex hull of three surjective isometries on* $C(\Omega)$, J. Math. Anal. Appl. 379 (2011), no. 2, 878–888.

Let I denote the identity operator on X, and $\mathbb{T}$ the unit circle in $\mathbb{C}$. Here the authors define the notion of generalized n-circular projections as follows:

Definition [1.0.5]. *A projection P_0 on a complex Banach space X is said to be a generalized n-circular projection (GnP, in short), $n \geq 2$, if there exist $\lambda_1, \lambda_2, \ldots, \lambda_{n-1} \in \mathbb{T} \setminus \{1\}$, λ_i of finite order, $i = 1, 2, \ldots, n-1$, and non-trivial projections $P_1, P_2, \ldots, P_{n-1}$ on X such that*

1. *$\lambda_i \neq \lambda_j$ for $i \neq j$,*
2. *$P_0 \oplus P_1 \oplus \cdots \oplus P_{n-1} = I$,*
3. *$P_0 + \lambda_1 P_1 \cdots + \lambda_{n-1} P_{n-1}$ is a surjective linear isometry on X.*

Every GnP is contractive, i.e., $\|P\| = 1$. The case $n = 2$ is known as generalized bi-circular projections (GBP). It is known that GBPs are bicontractive, i.e., $\|P\| = \|I - P\| = 1$, and on certain functions spaces, it was proved that any bicontractive projection is a GBP.

In this paper, the authors give a complete description of $G3P$ on $C(\Omega)$ where Ω is a compact connected Hausdorff space. The main result is:

[1.0.6]. *Let Ω be a compact connected Hausdorff space, and let P be a projection on $C(\Omega)$ such that $P = \alpha_1 T_1 + \alpha_2 T_2 + \alpha_3 T_3$, where T_1, T_2, T_3 are surjective isometries on $C(\Omega)$, $\alpha_1, \alpha_2, \alpha_3 > 0$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Then either*

1. *$\alpha_i = \frac{1}{2}$ for some $i = 1, 2, 3$, $\alpha_j + \alpha_k = \frac{1}{2}$, $j, k \neq i$ and $T_j = T_k$. In this case P is a GBP; or*

2. *$\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{3}$ and T_1, T_2, T_3 are distinct surjective isometries. In this case P is a G3P.*

This paper is also one of the five most cited papers by Sudipta.

[22] A. B. Abubaker, S. Dutta, *Structures of generalized 3-circular projections for symmetric norms*, Proc. Indian Acad. Sci. Math. Sci. 126 (2016), no. 2, 241–252.

In this paper, the authors characterize generalized 3-circular projections on $\mathbb{C}^n$ equipped with a symmetric norm and on $\mathbb{M}_{m\times n}(\mathbb{C})$, the space of all $m \times n$ complex matrices, equipped with a unitarily invariant norm.

[18] S. Dutta, Bor-Luh Lin, *Local U-convexity*, J. Convex Anal. 18 (2011), no. 3, 811–821. 46B20

A Banach space X is said to be a U-convex space [41] if for any $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $x, y \in S(X)$ with $\frac{1}{2}\|x + y\| > 1 - \delta$, we have $f(y) > 1 - \varepsilon$ for all $f \in D(x)$.

In this paper the authors study U-convexity and its localization quantitatively, through certain moduli. They define two possible localizations of U-convexity, namely $U_S(x, t)$ and $U_I(x, t)$ below.

Definition [1.0.7]. *For $x \in S(X)$, $f \in S(X^*)$ and $0 < t < 2$, let $S(x, f, t) = \{y \in B(X) : f(y) > f(x) - t\}$ and $S(x, f, t)^c = B(X) \setminus S(x, f, t)$. Define*

$$U(x, f, t) = \inf_{y\in S(x,f,t)^c} \{1 - \frac{1}{2}\|x + y\|\}$$

and define

$$\begin{array}{rclcrcl} U_S(x,t) & = & \sup_{f\in D(x)} U(x,f,t) & & U_I(x,t) & = & \inf_{f\in D(x)} U(x,f,t) \\ U_S(t) & = & \inf_{x\in S(X)} U_S(x,t) & & U_I(t) & = & \inf_{x\in S(X)} U_I(x,t). \end{array}$$

For U-convex spaces, $U_S(t) = U_I(t) > 0$ for all $t \in (0, 2)$. However, $U_S(x, t) \neq U_I(x, t)$ in general.

The authors show that if $x \in S(X)$ is a Fréchet smooth point, then $U_I(x, t) > 0$ for all $t \in (0, 2)$.

Let $x \in S(X)$ and $f \in D(x)$. The authors show that f is a LUR point if and only if x is a Fréchet smooth point and $U_I(f,t) > 0$ for all $t \in (0,2)$.

They also define the moduli of denting point $(d(x,t))$ and strongly exposed point $(s(x,t))$ and their uniform versions ($d(t)$ and $s(t)$) and show that

(a) x is a denting point if and only $d(x,t) > 0$ for all $t \in (0,2)$.

(b) x is a strongly exposed point if and only if $s(x,t) >$ for all $t \in (0,2)$.

(c) If $d(t) > 0$ for all $t \in (0,2)$, then X is superreflexive.

(d) If $s(t) > 0$ for all $t \in (0,2)$, then X is uniformly convex.

[19] S. Dutta, P. Mohanty, U. B. Tewari, *Multipliers which are not completely bounded*, Illinois J. Math. 56 (2012), no. 2, 571–578.

Let G be a locally compact abelian group and $1 \leq p < \infty$. A bounded linear operator $T : L^p(G) \to L^p(G)$ is said to be a $L^p(G)$ multiplier if commutes with translations. Denote the space of all multipliers by $M_p(G)$.

It follows from the work of G. Pisier [44] that $L^p(G)$ can be equipped with a natural operator space structure. If $T \in M_p(G)$ is completely bounded in the above operator space structure of $L^p(G)$, we call this a **cb-multiplier** on $L^p(G)$. Denote the space of all cb-multipliers on $L^p(G)$ by $M_p^{cb}(G)$. It is clear that $M_p^{cb}(G) \subseteq M_p(G)$. One can show that $M_p^{cb}(G) = M_p(G)$ for $p = 1, 2$. It is natural to ask: Is

$$M_p^{cb}(G) \subsetneq M_p(G) \text{ for } 1 < p \neq 2 < \infty? \tag{1.1}$$

Pisier also established that for G compact abelian and $1 < p < 2$, (1.1) holds. In this paper, the authors show that (1.1) holds for $\mathbb{R}$ and outline a proof for any group G infinite, locally compact and abelian. The main tool is a transference result for completely bounded multipliers.

[21] S. Dutta, P. Mohanty, *Completely bounded translation invariant operators on L_p*, Bull. Sci. Math. 139 (2015), no. 4, 420–430.

In this paper the authors define a Banach space $A_p^{cb}(G)$, and they show that, for abelian groups G, $A_p^{cb}(G)$ is an isometric pre-dual of $M_p^{cb}(G)$.

It is also proved that if G is a locally compact abelian group, $1 \leq p, q < \infty$ and $\left|\frac{1}{p} - \frac{1}{2}\right| < \left|\frac{1}{q} - \frac{1}{2}\right|$, then

$$M_q^{cb}(G) \subsetneq M_p^{cb}(G),$$

significantly improving on the result of [39].

Dr. Divya Khurana was Sudipta's second Ph. D. student.

[23] S. Dutta, Divya Khurana, *Ordinal indices of small subspaces of L_p*, Mediterr. J. Math. 13 (2016), no. 3, 1117–1125.

This paper focuses on the widely investigated problem of classifying (up to isomorphism) complemented subspaces of L_p spaces, $1 < p < \infty$, $p \neq 2$. Some "simple" examples of such spaces are L_p and ℓ_2, ℓ_p, $\ell_p \oplus \ell_2$ and $\ell_p(\ell_2)$.

In 1970, Rosenthal [46] constructed two other examples namely X_p and B_p, which are not isomorphic to any of the above-mentioned five spaces, and Alspach constructed a space D_p in the year 1974 (see [38]). In 1975, Schechtman [47] constructed countably many examples of complemented subspaces of L_p spaces, $1 < p < \infty$, $p \neq 2$, by taking repeated tensor product of the Rosenthal's space X_p with itself.

All these examples of complemented subspaces of L_p spaces, $1 < p < \infty$, $p \neq 2$, including the above mentioned class by Schechtman are subspaces of $\ell_p(\ell_2)$. This leads to two natural questions. First, can one go beyond the subspaces of $\ell_p(\ell_2)$ and second most importantly, are there uncountably many mutually non-isomorphic complemented subspaces of L_p spaces, $1 < p < \infty$, $p \neq 2$. The solution to both the questions finally came when J. Bourgain, H. Rosenthal and G. Schechtman [33] showed that there are uncountably many complemented subspaces of L_p, $1 < p < \infty$, $p \neq 2$, which are mutually non-isomorphic. In order to do so, they introduced the so-called ordinal h_p-index of a separable Banach space.

h_p-index is an isomorphic invariance for the class of separable Banach spaces. In [33] only a lower bound for the h_p-index of the constructed class of complemented subspaces of L_p spaces, $1 < p < \infty$, $p \neq 2$, was given. Natural questions which arise here are, is h_p-index an complete isomorphic invariance and can one compute h_p-index for subspaces of L_p explicitly.

In this paper, the authors find the ordinal h_p-index for Rosenthal's space X_p, ℓ_p and ℓ_2, $2 < p < \infty$. It is proved that for any infinite dimensional subspace of L_p, $2 < p < \infty$, possible values of h_p-index are ω_0, $\omega_0 \cdot 2$ or greater than equal to ω_0^2. As an application to this result it is proved that any infinite dimensional subspace of L_p, $2 < p < \infty$, which is not isomorphic to ℓ_2 embeds in ℓ_p if and only if its ordinal index is the minimal possible.

Dr. Aryaman Sensarma was a Ph. D. student under Sudipta at the time of his demise. Aryaman eventually completed his Ph. D. in 2019 under the joint supervision of Prof. Pradipta Bandyopadhyay of ISI Kolkata and Prof. Sameer Chavan of IIT Kanpur.

[27] S. Dutta, D. Khurana, A. Sensarma, *Representing matrices, M-ideals and Tensor products of L_1-predual spaces*, Extracta Math. 33 (2018), no. 1, 33–50.

A. J. Lazar and J. Lindenstrauss [43] showed that separable L_1-preduals can be represented by countable triangular matrices. In this paper, the authors define a diagrammatic representation of separable L_1-predual spaces and study the question whether the M-ideals in such a space have a diagrammatic representation based on the matrix representation of the space. Their motivation comes from the fact that separable L_1-preduals can be viewed as an isometric version of approximately finite-dimensional real C^*-algebras and that in such an algebra, the ideals are in a one-to-one correspondence with the directed sub-diagram of its Bratelli diagram. The authors show that, indeed, in complete analogy, in such spaces, every directed sub-diagram represents an M-ideal. The converse, namely, the question whether given an M-ideal in a separable L_1-predual space X, there exists a diagrammatic representation of X such that the M-ideal is given by a directed sub-diagram, remains open in general. We refer to this as "the main problem". In this paper, the authors partially answer the question by proving that this holds for $C([0,1])$ as well as for the spaces $C(K)$ where K is a totally disconnected compact metric space.

The authors also provide an algorithm for finding a representing matrix for the injective tensor product of two separable L_1-preduals having at hand representing matrices of the two spaces.

[28] P. Bandyopadhyay, S. Dutta, A. Sensarma, *Polyhedral preduals of ℓ_1 and their representing matrices*, J. Math. Anal. Appl. 468 (2018), no. 2, 1082–1089.

As we saw above, the isometric structures of separable real L_1-predual spaces are determined by their representing matrices.

In this paper, the authors are interested in the following question: Given a representing matrix A of a separable L_1-predual space X, is it possible to identify some properties of A that will ensure $X^* = \ell_1$?

Among separable L_1-predual spaces, one property that distinguishes X as an ℓ_1-predual space is polyhedrality (that is, to be a Banach space such that the unit ball of every finite-dimensional subspace is a polytope). The notions of polyhedrality are classified in four "categories".

The authors recall all notions mentioned above and analyze some relations among them. In particular, they obtain characterizations of three of the four "categories" of polyhedrality in a separable L_1-predual space in terms of its representing matrix.

The authors also show that in a polyhedral (IV) predual of L_1, their "main problem" has an affirmative answer.

[26] P. Bandyopadhyay, S. Dutta, A. Sensarma, *Almost isometric ideals and non-separable Gurariy spaces*, J. Math. Anal. Appl. 462 (2018), no. 1, 279–284.

This paper is on non-separable Gurariy spaces and their almost isometric ideals. The main result is that a (non-separable) Banach space is a Gurariy space if and only if every separable *almost isometric ideal* (a.i.-ideal) in X is isometric to the separable Gurariy space $\mathbb{G}$. Further, it is shown that a Banach space is an L_1-predual if and only if every separable ideal in it is an L_1-predual. Along the way, they show that the family of ideals/a.i.-ideals in a Banach space is closed under increasing limits. And hence, the family of all separable ideals/a.i.-ideals in a Banach space is a skeleton.

[29] S. Dutta, C. R. Jayanarayanan, Divya Khurana, *Ideal operators and relative Godum sets*, Extracta Math. 34 (2019), no. 1, 1–17.

Motivated by the principle of local reflexivity, G. Godefroy, N. J. Kalton and P. D. Saphar [37] introduced the notion of an ideal in Banach spaces. In this paper, the authors show that Y is an ideal in X if and only if there is an operator $T : X \to Y^{**}$ such that $\|T\| \leq 1$ and $T|_Y = Id_Y$, providing another point of view on ideals. Authors call such operator T an ideal operator. Motivated by this characterization, the authors have studied the variants of ideals, such as strict ideals, u-ideals, h-ideals or a.i.-ideals in terms of the ideal operator. In particular, authors have characterized a.i.-ideals. Mainly, uniqueness or injectivity of ideal operators seems to be crucial to reflect properties of ideals and so they are widely discussed. Finally, there are examples to illustrate how such a point of view may be applied.

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Chapter 2

A survey of Pravir Dutt's research contributions

Pravir Dutt, born in 1958 and passed away in 2021, graduated from the Don Bosco School in Kolkata in 1975 and earned an M.Sc. (Five Year Integrated) degree from the Indian Institute of Technology (IIT), Kanpur, in 1980. He earned a Ph.D. in 1985 from the University of California, Los Angeles, in the United States. In 1987, Pravir joined IIT Kanpur as a faculty member in the Department of Mathematics (now renamed as Department of Mathematics and Statistics) following a brief stay (1985–1987) at the NASA Langley Research Center, Hampton, Virginia, USA. In 2001, he was elevated to the position of Professor.

Pravir was an exceptional researcher known for his outstanding contributions to fluid mechanics, parallel computing, spectral element methods, and numerical solutions of partial differential equations (PDEs). Under his guidance, more than a dozen Ph.D. candidates in various areas of Applied Mathematics completed their theses.

Developing spectral element approaches for the numerical solution of elliptic, parabolic, and hyperbolic problems is one of his many significant achievements. In addition to offering theoretical frameworks and associated stability and convergence analyses, his research into numerical methods also involved practical implementations, frequently on parallel computing infrastructure.

In the sections that follow, we provide a summary of his contributions to the numerical solution of partial differential equations and his research accomplishments in fluid mechanics.

2.1 Spectral methods for initial-boundary value problems for hyperbolic systems of partial differential equations

Pravir Dutt examined several facets of numerical solutions of initial-boundary value problems for hyperbolic partial differential equation systems. Specifically, he developed and analyzed spectral approaches for solving such problems.

For instance, in [1], Dutt proved the stability of a regularized spectral method for hyperbolic systems on a finite interval. The regularization occurs at boundaries and amounts to a convolution of the outgoing waves with a smoothing function. This procedure avoids any loss of smoothness upon reflection.

In a related follow-up work [2], he proposed a new approach to spectral methods for the numerical solution of initial-boundary value problems for hyperbolic systems of partial differential equations. In this scheme, Chebyshev polynomials are chosen as a basis for an approximate solution. The unknowns are determined to minimize a certain weighted average of the residuals corresponding to the given partial differential equations and the initial and boundary conditions.

He also contributed toward the numerical solution of the initial-boundary value problems with non-smooth data. To begin with, in [3], he examined hyperbolic initial value problems with periodic but not necessarily smooth data. In this paper, authors showed that if they filter the data and solve the problem using their Galerkin-Collocation method, they can recover pointwise values with spectral accuracy, provided that the solution is piecewise smooth. They relied on a local smoothing of the computed solution to achieve this. They also proved error estimates in Sobolev norms of negative order with respect to space and time. In his 1999 Numerische Mathematik paper [4], he extended these ideas to solve the initial-boundary value problems with non-smooth data. He showed that if we extend the time domain to minus infinity, replace the initial condition by a growth condition at minus infinity and then solve the problem using a filtered version of the data by the Galerkin-Collocation method using Laguerre polynomials in time and Legendre polynomials in space, then we can recover pointwise values with spectral accuracy, provided that the actual solution is piecewise smooth. For this, he again performed a local smoothing of the computed solution.

2.2 Spectral element methods for elliptic problems

Another prominent direction of research contributions of Pravir Dutt is in developing the spectral element method for elliptic problems.

The spectral element methods exhibit exponential convergence for smooth problems and have been successfully used in practical situations. However, we frequently need to numerically solve boundary value problems in non-smooth domains in many engineering and scientific applications. It is well known that the solutions of elliptic boundary value problems have singular behavior near the corners and edges of the domain. Due to the presence of singularities, conventional numerical methods fail to provide accurate numerical solutions, and the rate of convergence of these methods degrades. In such cases, they offer no advantages over low-order methods. It is desirable to find efficient and robust methods along with standard numerical techniques to improve the accuracy of the solutions and the efficiency of computations.

In this context, Dutt et al. proposed [5] a new parallel h-p spectral element method that resolves the underlying singularities by employing a geometric mesh in the neighborhood of the corners. This scheme gives exponential convergence with asymptotically faster results than conventional methods. They also derived relevant stability estimates for this approach, first in polygonal domains [6], and then for general elliptic problems on curvilinear domains [7]. A non-conformal version of this method was also proposed by Dutt et al. in [8].

In a sequence of papers [9, 10, 11], Dutt et al. proposed and analyzed a nonconforming h-p spectral element method for 3D elliptic boundary value problems on non-smooth domains. The procedure uses auxiliary mappings and geometrical mesh refinement to resolve the singularities at an exponential rate in anisotropic weighted Sobolev spaces with respect to the number of layers in the mesh and polynomial order used in approximation. The method is essentially a least-squares method, and the preconditioned conjugate gradient method (PCGM) is used to solve the normal equations arising from the least-squares formulation. The scheme is implemented on a parallel computer with distributed memory where the library used for message passing is MPI. The numerical results confirm the theoretical estimates.

2.3 Spectral element methods for parabolic problems

Pravir Dutt also contributed to the numerical solution of parabolic problems. This work presents a spectral method for solving parabolic initial boundary value problems on smooth domains using parallel computers. Dutt et al. minimize, at each time step, a functional which is the sum of the squares of the residuals in the partial differential equation, initial condition and boundary condition in different Sobolev norms and a term which measures the jump in the function and its derivatives across inter-element boundaries in a certain Sobolev norm. The Sobolev spaces used are of different orders in time and space. Error estimates are obtained for both the h and p versions of this method.

In [13], a non-conforming least-squares spectral element method for parabolic initial value problems with non-smooth data is introduced. The method converges exponentially in space and time and minimizes residuals in the partial differential equation and initial condition in different Sobolev norms. Continuity is enforced by adding a term to the function being minimized, which measures the jump in the function and its derivatives across inter-element boundaries in appropriate fractional Sobolev norms. The difficulty associated with the non-smooth initial data is resolved using the Hermite mollifier described by E. Tadmor and by J. Tanner. Parallelization and preconditioning of the method are described, and rigorous error estimates are derived. Several numerical examples are provided to demonstrate the accuracy and efficiency of the method. One of the examples is the European Black- Scholes Rainbow Put Options problem. Results for this problem with the proposed method are found to be superior to the results obtained by W. Zhu and D. A. Kopriva and by M. J. Ruijter and C. W. Oosterlee.

2.4 Fluid Mechanics

In [14], Pravir Dutt studied stable boundary conditions and difference schemes for Navier-Stokes equations. More precisely, by employing the entropy function for the Euler equations as a measure of "energy" for the Navier-Stokes equations, he obtained nonlinear "energy" estimates for the mixed initial boundary value problem. These estimates are then used to derive boundary conditions that guarantee L^2 bounded solutions with weak boundary layers even as the Reynolds number tends to infinity. In the same paper, he proposed a new difference scheme for modeling the Navier-Stokes equations in multidimensions. He obtained discrete energy estimates exactly analogous to those he derived for the differential equation.

Another notable contribution in this direction was published recently, where he, along with his collaborators, proposed a non-conforming least-squares spectral element method for Stokes equations [15]. More precisely, this work presents a discontinuous least-squares spectral element method for Stokes equations with primitive variable formulation on both smooth and non-smooth domains. The proposed numerical scheme is based on stability estimates and is exponentially accurate. The method was implemented on different polygonal domains to demonstrate its efficacy in terms of accuracy.

More recently, he contributed to a study of granular flow on a rotating and gravitating elliptical body [16]. This work investigates two-dimensional shallow granular flows on a rotating and gravitating elliptical body. This is motivated by regolith flow on small planetary bodies - also called minor planets - which is influenced by the rotation of the body, as well as its irregular topography and complex gravity field.

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Chapter 3

A small highlight of research work of Professor Arbind K. Lal

Abstract: *This document is prepared with contributing write-ups from different coauthors of Professor Arbind Kumar Lal. It aims to highlight some of his research works. Owing to Professor Lal's ability to work in different areas and the fact that we could not collect contributions from all of his coauthors, this document is nowhere close to completely covering his entire research carrier.*

3.1 Professor Arbind Lal

Professor Arbind Lal's coauthors and students will always remember him as being an enthusiastic person who enjoyed engaging in mathematical discussions with them. In addition to this, he was a very kind and helpful person.

He completed his PhD from Indian Statistical Institute, New Delhi, under the supervision of Prof. R. B. Bapat in 1993. After brief stints at TIFR Mumbai and HRI Allahabad, Arbind joined IIT Kanpur, as a faculty in 1996.

In this article we will discuss some of his mathematical contributions. His work exhibits a great deal of diversity. This collection is only a small part of his contributions, which is far from the complete picture.

According to his students, he was instrumental in changing their philosophy of life. He also influenced their work culture in general and their mathematics in particular.

Here, we have tried to keep mathematical discussions accessible in order to highlight the fundamental importance of his works.

3.2 Algebraic connectivity

Have you ever wondered which city is more connected out of two given cities? We start to imagine some localities in cities and key to have an overall idea of how effectively we can go from one place to another via their transport system.

In a picture, let us represent the localities by dots (which we call vertices). We join a pair of vertices by a line segment (which we call an edge) and allow curves, if there is a direct transport available to travel between those two places.

What we obtain here is called a graph. The edges here could be weighted based on different factors like time taken for travel or modes available etc. With this model in mind, one then asks, which graph is better connected?

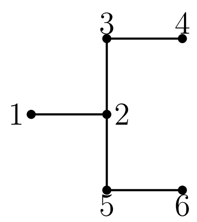
Imagine for simplicity that all the edges have weight 1. Even in this situation, the answer is not easy. One of the many ways to answer this question is to use the algebraic connectivity.

For this one needs to label the vertices ase $1, 2, \ldots, n$. We use G to denote a graph with $V = \{1, \ldots, n\}$ to be the vertices and E to be the edges (this is a set of some unordered pairs of distinct vertices). Two vertices are called adjacent, if there is an edge available between them.

Corresponding to the graph G, we can associate the *adjacency matrix* A whose (i, j)-th entry

$$a_{ij} = \begin{cases} 1 & \text{if } i \sim j \text{ (adjacent)} \\ 0 & \text{otherwise.} \end{cases}$$

We also associate the *Laplacian matrix* L which is defined by $L := D - A$, where D is the diagonal degree matrix. Here the degree of a vertex i is the number of edges incident on it.



$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

G $\qquad$ A $\qquad$ L

Professor Mirslav Fiedler, in 1973, observed that the matrix L has many special properties. For example, it is always a *positive semidefinite matrix* and therefore it has all its eigenvalues being nonnegative. The smallest eigenvalue is always 0.

He observed that if the graph is *connected* (travel between each pair of vertices is possible), then the second smallest eigenvalue is always positive and if we add more edges to the graph, this number only increases.

Hence, this number may be viewed as a quantity that measures the connectivity of the graph and is called the *algebraic connectivity* of G.

Many natural results followed. For example, it was shown that the complete graph K_n (where all possible edges exist between the vertices) has the highest algebraic connectivity which is n. Further, among the connected graphs, the *path* P_n on n vertices (here $[1, 2], [2, 3], \ldots, [n-1, n]$ are the only edges) has the smallest algebraic connectivity.

Many researchers contributed to the study. It was shown that an eigenvector corresponding to the algebraic connectivity identifies a somewhat central part of the graph called the *characteristic set.*

For very special classes of graphs called *trees* (these are connected graphs in which cycles cannot be found), it was also proved that if we try to move the edges away from the characteristic set, the algebraic connectivity tends to decrease.

We illustrate this in the following example. In figure 2.1, the tree T is not completely shown. The characteristic set is assumed to be on the dotted side to the left of u. The curved segment is not part of the tree. Observe the part to the right of the curved segment. It is called a branch B at u. Observe, how the edge X moves away one step from u and we obtain the tree T' on the right. It is known that the algebraic connectivity decreases in this process.

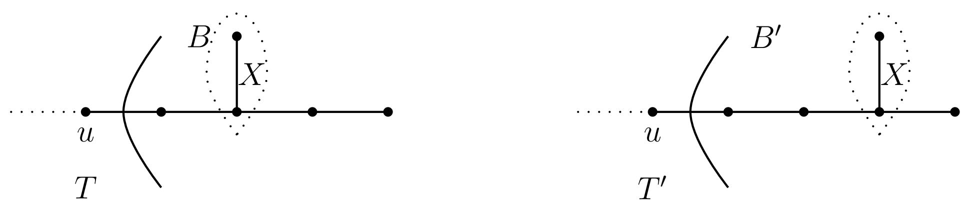


Figure 3.1: The algebraic connectivity $a(T') \leq a(T)$.

A more general result was always believed to be true but was never proved. For example, consider the following graph G (figure 2.2). It may be having more vertices on the left side. Here, we assume the characteristic set is in the left side of 1. Consider the path $P = [2, 3, 4, 5, 6, 7]$ and the structure X which is inside the dotted curve. The graph H is now obtained by moving X one unit away from 1 along the path P.

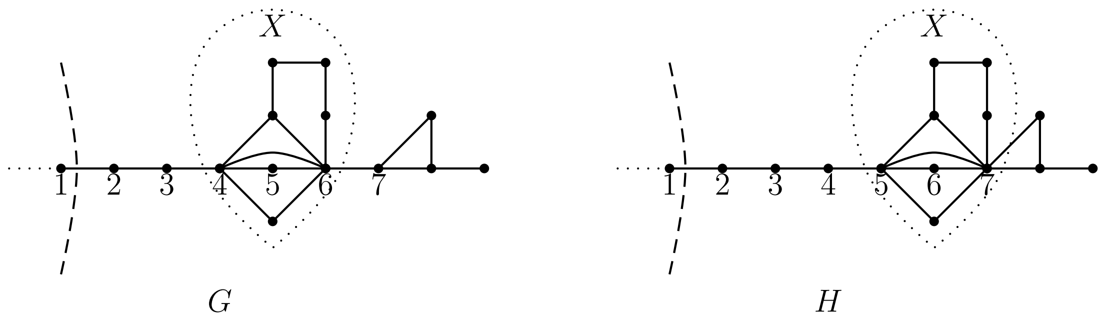
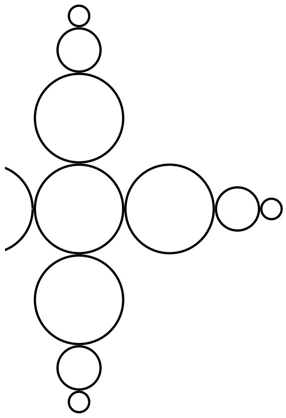


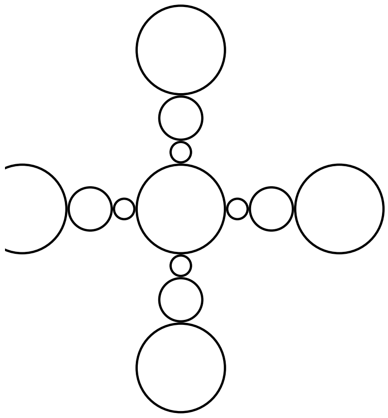
Figure 3.2: H is obtained from G by sliding C along P one unit away from $u = 1$.

In such a situation one would ask whether $a(H) \leq a(G)$. Professor Lal and his coauthors have shown that the result is indeed true.

Not much is known for graphs with more complicated structure. However, the following notable comparison is a contribution of Professor Lal and his coauthors. Consider a graph G with the structure given in the left side of the following picture. Imagine rearranging the structures in the four sides to get H. Can we say that $a(H) \leq a(G)$?



G



H

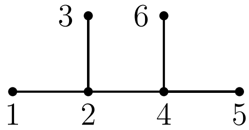
One can imagine rotating them about their centers and getting a feeling as if the graph H will fall apart first. This sense is captured by their algebraic connectivities. So the answer is yes. It has been shown by Professor Lal and his coauthors that $a(H) \leq a(G)$.

For more information, please refer to 'On algebraic connectivity of graphs with at most two points of articulations in each block, *Linear Multilinear Algebra*, 60(4), (2012), 415–432' and the references therein.

Some material here discusses results from a recently finalized and submitted article by Professor Lal and his coauthors, titled 'Some observations on algebraic connectivity of graphs'.

3.3 Distance matrix

Consider a tree T. The distance matrix is the matrix $D(T)$ with its (i, j)-th entry equal to the distance between the vertices i and j. For example, for the tree in the following picture, the distance matrix is given below.



$$\begin{bmatrix} 0 & 1 & 2 & 2 & 3 & 3 \\ 1 & 0 & 1 & 1 & 2 & 2 \\ 2 & 1 & 0 & 2 & 3 & 3 \\ 2 & 1 & 2 & 0 & 1 & 1 \\ 3 & 2 & 3 & 1 & 0 & 2 \\ 3 & 2 & 3 & 1 & 2 & 0 \end{bmatrix}$$

A formula for the determinant of the distance matrix of a tree was supplied by Graham and Pollack in 1971.

[3.3.1]. *Theorem. Let T be a tree on vertices $1, 2, \ldots, n$. Then the determinant of the distance matrix is* $\det D(T) = (-1)^{n-1} 2^{n-2} (n-1)$.

This result tells us that the determinant of the distance matrix of a tree depends only on the number of vertices and it does not depend on the structure of the tree. The formula for the inverse of the matrix $D(T)$ was obtained in a subsequent paper by Graham and Lovasz.

There have been many generalizations of these result. One beautiful and useful generalization was give by Professor Lal and his coauthors.

They considered something called the q-distance matrix $\mathcal{D}$ whose (i, j)-th entry is defined as

$$\mathcal{D}_{ij} = \begin{cases} 1 + q + q^2 + \cdots + q^{k-1} & \text{if the usual distance between } i \text{ and } j \text{ is } k \\ 0 & \text{if } i = j. \end{cases}$$

Observe that, when we take $q = 1$, we obtain nothing but the usual distance matrix. For example for the previous tree the q-distance matrix is

$$\mathcal{D} = \begin{bmatrix} 0 & 1 & 1+q & 1+q & 1+q+q^2 & 1+q+q^2 \\ 1 & 0 & 1 & 1 & 1+q & 1+q \\ 1+q & 1 & 0 & 1+q & 1+q+q^2 & 1+q+q^2 \\ 1+q & 1 & 1+q & 0 & 1 & 1 \\ 1+q+q^2 & 1+q & 1+q+q^2 & 1 & 0 & 1+q \\ 1+q+q^2 & 1+q & 1+q+q^2 & 1 & 1+q & 0 \end{bmatrix}.$$

The following is one of the many striking results proved by them.

[3.3.2]. *Theorem. Let T be a tree on vertices $1, 2, \ldots, n$. Then the determinant of the distance matrix is* $\det D(T) = (-1)^{n-1} (1+q)^{n-2} (n-1)$.

The inverse of the q-distance matrix was also supplied. Many other variations like, weighted q-distance matrix, exponential distance matrix, were also studied by them. Since then, many articles has been published taking their study even further.

For more information, please refer to 'A q-analogue of the distance matrix of a tree, *Linear Algebra and its Applications*, 416 (2006) 799–814' and the references therein.

3.4 More works on the Laplacian matrix

Let G be a simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E. The adjacency matrix $A(G)$ of G is defined as $A(G) = [a_{ij}]$, where $a_{ij} = 1$ if $\{v_i, v_j\}$ is an edge of G and 0 otherwise. Let $D(G)$ be the diagonal matrix of vertex degrees of G. The Laplacian matrix of G is defined as $L(G) = D(G) - A(G)$. It is easy to see that $L(G)$ is a symmetric positive semidefinite matrix with 0 as an eigenvalue. Let the eigenvalues of $L(G)$ be $0 = \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$ repeated according to their multiplicity. Then $\lambda_2(G)$ and $\lambda_n(G)$ are known as algebraic connectivity and Laplacian spectral radius of G and we denote them by $\mu(G)$ and $\lambda(G)$, respectively. It is also known that $\mu(G) = 0$ if and only if G is connected and $\mu(G) = n - \lambda(\overline{G})$, where $\overline{G}$ is the complement of G. We mainly worked on some extremal problems associated with both $\mu(G)$ and $\lambda(G)$.

We study the effect on the algebraic connectivity of a tree by the grafting and collapsing of edges. As a corollary to these results, we prove that among all trees on n vertices, the path has the smallest and star has the largest algebraic connectivity.

Let $\mathcal{H}_{n,k}$ denote the set of all connected graphs on n vertices with k pendant vertices. It is known that the complete graph K_n has the maximum algebraic connectivity over $\mathcal{H}_{n,0}$. For $n \geq 6$, take a path on $n-4$ vertices and identify each pendant vertex with a vertex of K_3. We denote it by $C_{3,3}^{n-6}$. We prove that for $n \geq 6$, the graph $C_{3,3}^{n-6}$ uniquely attains the minimum algebraic connectivity over $\mathcal{H}_{n,0}$.

Let $1 \leq k \leq n-1$ and $n \geq 4$. For $k \neq n-2$, the graph P_n^k is obtained by adding k pendant vertices adjacent to a single vertex of the complete graph K_{n-k} and for $k = n-2$, the graph P_n^{n-2} is obtained by adding $n-3$ pendant vertices adjacent to a pendant vertex of the path P_3. We prove that the graph P_n^k attains the maximum algebraic connectivity over $\mathcal{H}_{n,k}$.

Let $T(q, l, d)$ be the tree obtained by taking a path P_d with end vertices v and w, and adding q pendant vertices adjacent to v and l pendant vertices adjacent to w. Let C_3^{n-3} be the graph obtained by identifying a vertex of K_3

with an end vertex of P_{n-2}. We prove that for $k \geq 2$, the tree $T(\lceil \frac{k}{2} \rceil, \lfloor \frac{k}{2} \rfloor, n-k)$ uniquely attains the minimum algebraic connectivity over $\mathcal{H}_{n,k}$ and C_3^{n-3} uniquely attains the minimum algebraic connectivity over $\mathcal{H}_{n,1}$. We also prove that over all unicyclic graphs on n vertices, the algebraic connectivity is uniquely minimized by the graph C_3^{n-3}.

We have also worked on some problem related to Laplacian spectral radius of trees. We prove that over all trees on n vertices with diameter d, the maximum Laplacian spectral radius is achieved uniquely at $T_{n,d}$, where $T_{n,d}$ is obtained by taking a path P on $d+1$ vertices and adding $n-d-1$ pendant vertices to a central vertex of P.

3.5 Partial differential equations

In the last three decades, a lot of work on the interaction between seemingly distant fields of mathematics, i.e., graph theory and analysis, has been done. This interaction has indeed contributed some interesting results in the theory of mathematical physics, probability theory, ergodic theory, harmonic analysis, partial differential equations etc. Some of the works concentrated on the study of Partial Differential Equations (PDEs).

The combinatorial Laplacian operator on graphs may be viewed as a discrete analogue of the classical Laplacian operator. Therefore, the combinatorial Laplacian operator and the problems related to the combinatorial PDEs on graphs find importance in the study of "analysis on graphs". The techniques that have been used in the study of analysis on graphs come from a very wide range of topics: algebra, combinatorics, PDEs, linear algebra, spectral theory, analysis etc. We have dealt with a few problems related to combinatorial PDEs on graphs.

Notations and Definitions: Let $G = (V, E)$ be a locally finite graph. For $1 \leq p \leq \infty$, let us consider the normed linear space $L^p(V) = \{\ f : V \to \mathcal{C} :\ \|f\|_{L^p(V)} < \infty\}$, where

$$\|f\|_{L^p(V)} = \begin{cases} \left(\sum\limits_{x \in V} |f(x)|^p \right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty, \\ \sup\limits_{x \in V} |f(x)|, & \text{if } p = \infty. \end{cases}$$

Also, for a fixed function $f : V \to \mathcal{C}$, the combinatorial Laplacian operator on G evaluated at f is defined as

$$\Delta_G f(x) \ = \sum_{y \sim x} (f(x) - f(y)) = \ m(x) f(x) - \sum_{y \sim x} f(y) \text{ for each } x \in V,$$

where $m(x)$ denote the degree of the vertex x. Note that Δ_G is bounded on $L^2(V)$ if and only if $m(x)$ is uniformly bounded.

The combinatorial Schrödinger operator on a graph G has the form $\mathbb{L}_q(G) = \Delta_G + q$, where q (called the potential) is a real valued function on the vertex set V of G. We restrict ourselves to nonnegative potentials, *i.e.*, $q \geq 0$.

For any proper subset S of V, the boundary of S, denoted ∂S, is defined as

$$\partial S = \{x \in V(G) \setminus S \ : \ \exists\, y \in S \ \text{ such that } x \sim y\}.$$

We denote the induced subgraphs on the vertex sets S and $\overline{S} = S \cup \partial S$ by $X[S]$ and $\overline{X}[S]$ (in short, X and $\overline{X}$), respectively. The vertex set S is chosen so that $|S|$ is finite and the induced subgraph X is connected. The graph $\overline{X}$ has S as its interior and ∂S as its boundary. We are interested in the Schrödinger eigenvalue problem with Dirichlet boundary condition with respect to the finite vertex set S is equivalent to the study of $\mathbb{L}_q(\overline{X}) = \Delta_{\overline{X}} + q$ with Dirichlet boundary condition. So, we are interested in the eigenvalue problem

$$\begin{cases} \Delta_{\overline{X}} f + qf &= \lambda(q) f \quad \text{in } S, \\ f &= 0 \quad\quad\;\; \text{on } \partial S. \end{cases}$$

Before proceeding further, we need to define the difference operator on the set of non-negative integers which is the discrete analogue of differentiation with respect to the time variable.

Definition. *Let* $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. *Then, for any complex valued function* $v : \mathbb{Z}_+ \to \mathcal{C}$, *we define* $\partial_n v(n) = v(n+1) - v(n)$.

Problems of Interest and Related Outputs: With the definitions and notations as above, we state the two problems in combinatorial PDEs and the related outputs.

1. Finding the extrema of the eigenvalue problems related to combinatorial PDEs that are analogues of classical PDEs. To be precise, we are interested in maximizing the smallest eigenvalue $\lambda_1(q)$ of the combinatorial Schrödinger operator $\mathbb{L}_q(\overline{X})$ with Dirichlet boundary condition whenever the non-negative potential q lies in the unit disc of L^p.

   Output: For a nonnegative potential function q and a given graph G, we study the combinatorial Schrödinger operator $L_q(G) = \Delta_G + q$ with Dirichlet boundary condition. Let S be a proper finite subset of the vertex set of G such that the induced subgraph on S is connected and let $\Upsilon_p = \{q \in L^p(S) \ : \ q(x) \geq 0, \ \sum_{x \in S} q^p(x) \leq 1\}$, for $1 \leq p < \infty$. We prove the existence and uniqueness of the maximizer of the smallest Dirichlet eigenvalue of $L_q(G)$, whenever the potential function $q \in \Upsilon_p$. Furthermore, we also establish the analogue of the Euler-Lagrange equations on graphs.

2. Studying the solutions of the combinatorial heat and wave equations on certain classes of graphs, viz., the Cayley and coset graphs (these graphs are defined in later part of the synopsis). To be precise, given a Cayley or a coset graph G, we want to solve the combinatorial heat equation

$$\begin{aligned} \Delta_G u(x,n) + \partial_n u(x,n) &= 0 \text{ on } V(G) \times \mathbb{Z}_+, \\ u(x,0) &= f(x), \end{aligned}$$

and the combinatorial wave equation

$$\begin{aligned} \Delta_G u(x,n) + \partial_n^2 u(x,n) &= 0 \text{ on } V(G) \times \mathbb{Z}_+, \\ u(x,0) &= f(x), \quad \partial_n u(x,0) = g(x). \end{aligned}$$

Output: • Given any finite Cayley graph G, the solution to the combinatorial heat equation is of the form $u(x,n) = K_n * f(x)$ and the solution to the combinatorial wave equation is of the form $u(x,n) = F_n * f(x) + G_n * g(x)$. It is interesting to note that these solutions are in the form of a convolution and are similar to the solutions in the classical case.

• Using the understanding built during the process of solving the heat and wave equations for finite Cayley graphs, we are also able to solve these equations on finite coset graphs. Thus, our technique helps us in solving the combinatorial heat and wave equations on all finite vertex transitive graphs. Further, using the theory of Fourier analysis on locally compact abelian groups, we are also able to extend our results to certain classes of infinite Cayley and coset graphs.

• We are also able to solve the combinatorial heat and wave equations on k-regular trees, which can be identified with an infinite Cayley graph G, where the associated group is a non-abelian free group with generators $s_1, s_2, ..., s_k$, each of order 2.

3.6 Symmetry breaking in graphs

We distinguish or identify the pages of a book by its page numbers. Imagine a book in which the pages are not numbered. Rather, some shapes are attached to the corners of the pages so that the pages can still be identified by mere touching and feeling of the shapes. If the number of shapes are equal to the number of pages and each page is assigned a different shape, then clearly all the pages can be identified by touching and feeling the shapes. However, this can be done with fewer number of shapes too. For example, if a book contains only four pages, then only two shapes are enough to identify its pages. See Figure 3.3(i), in which we use the labels a and b to denote two distinct shapes. Observe that the corners of the four pages have been labeled by the four distinct strings of labels $aaab, aabb, abab$ and $abbb$. Hence all the four pages can be distinguished. Similarly, if the number of pages of a book is between 5 and 9, then this can be done with only three labels.

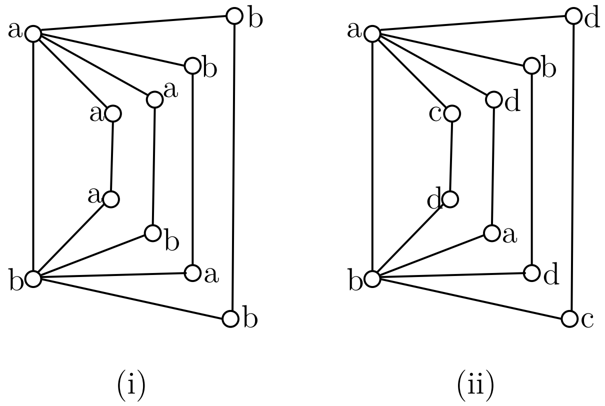


Figure 3.3: The graph $B_{4,4}$.

Note that due to several symmetries of a book, its pages cannot be identified without any page numbers, shapes or labels attached to the pages. For example, all permutations of the pages as well as a flip about a line perpendicular to the spine are symmetries of a book. One can notice that there is no nontrivial label preserving symmetry for the book in Figure 3.3(i). In general, if the corners of the pages of a book are labeled in such a way that there remains no nontrivial label preserving symmetry of the book, then all the pages of the book can be identified uniquely. Thus, given a book with n pages, we wish to determine the minimum number of labels needed to label the corners of the pages so that no nontrivial label preserving symmetry of the book survives. That is, all the nontrivial symmetries of the book are destroyed or broken by such a labeling. Normally, the pages of a book are rectangular in shape. We generalize this idea and consider a book in which the pages are polygonal in shape. For $m \geq 3$ and $n \geq 1$, we use the notation $B_{m,n}$ to denote a book with m pages in which all the pages are regular n-gons. Indeed, $B_{m,n}$ is a simple connected graph. In general, we wish to determine the minimum number r for a simple connected graph G such that the vertices of G can be labeled with r labels and that no nontrivial label preserving symmetry, or rather automorphism, of the graph survives. We denote this minimum number by $D(G)$, and call it the *distinguishing number* of G. The distinguishing number of $B_{m,n}$ is determined by Professor Lal and one of his coauthors, as given in the following formula:

$$D(B_{m,n}) = \begin{cases} 3 & \text{if } m \in \{3,4,5\},\ n=1 \\ 2 & \text{if } m \geq 6,\ n=1 \\ n & \text{if } m=3,\ n \geq 2 \\ k & \text{if } (k-1)^{m-2}+1 \leq n \leq k^{m-2},\ k \geq 2,\ m \geq 4,\ n \geq 2. \end{cases}$$

The distinguishing number of a graph can be made more refined if we assume the labeling to satisfy the condition that adjacent vertices receive distinct labels. Such a distinguishing number of a graph G is called the *distinguishing chromatic number* of G, and it is denoted by $\chi_D(G)$. For example, $\chi_D(B_{4,4}) = 4$. In Figure 3.3(ii), a labeling of $B_{4,4}$ is given with four labels such that adjacent vertices receive distinct labels and no label preserving nontrivial automorphism of $B_{4,4}$ survives.

The distinguishing chromatic number of $B_{m,n}$ is determined in a work of Professor Lal and one of his coauthors, as given in the following formula:

$$\chi_D(B_{m,n}) = \begin{cases} 4 & \text{if } m \in \{4,6\}, n=1 \\ 3 & \text{if } m \in \{3,5\} \cup \{7,8,\ldots\}, n=1 \\ n+2 & \text{if } m=3, n \geq 2 \\ k & \text{if } \alpha(m,k-1)+1 \leq n \leq \alpha(m,k),\ k \geq 3,\ m \geq 4,\ n \geq 2, \end{cases}$$

where $\alpha(m,k) = \frac{1}{2}(k-1)^{\frac{m-4}{2}}[1+(-1)^m] + (k-2)(k-1)^{m-3}$.

Let n and k be positive integers such that $2 \leq 2k < n$. The *generalized Petersen graph*, denoted $P_{n,k}$, is defined to have the vertex set

$$V(P_{n,k}) = \{u_0, \ldots, u_{n-1}\} \cup \{v_0, \ldots, v_{n-1}\}$$

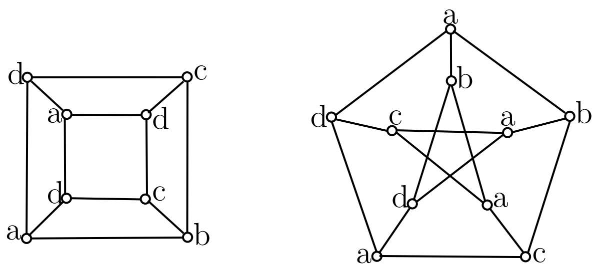
and the edge set

$$E(P_{n,k}) = \{u_i u_{i+1}, v_i v_{i+k}, u_i v_i \ :\ i \in \{0,1,\ldots,n-1\}\},$$

where the subscripts are read modulo n. The graphs $P_{4,1}$ and $P_{5,2}$ are shown in Figure 3.4.

The number $D(P_{n,k})$ was determined by other researchers as follows:

$$D(P_{n,k}) = \begin{cases} 3 & \text{if } (n,k)=(4,1) \text{ or } (n,k)=(5,2) \\ 2 & \text{otherwise.} \end{cases}$$



(i) $P_{4,1}$ (ii) $P_{5,2}$

Figure 3.4: The graphs $P_{4,1}$ and $P_{5,2}$.

In Figure 3.4, a labeling of each of $P_{4,1}$ and $P_{5,2}$ are given with four labels such that adjacent vertices receive distinct labels and no label preserving nontrivial automorphism of the corresponding graphs survive. In a work by Professor Lal and one of his coauthors, the number $\chi_D\left(P_{n,k}\right)$ was determined as follows:

$$\chi_D\left(P_{n,k}\right) = \begin{cases} 4 & \text{if } (n,k) = (4,1) \text{ or } (n,k) = (5,2) \\ 3 & \text{otherwise.} \end{cases}$$

Interestingly, $\chi_D\left(P_{n,k}\right) = D\left(P_{n,k}\right) + 1$ for each n and k. However, this is not the case in general. For example, $D(P) = 2 = \chi_D(P)$, where P is a path on even number of vertices. Also, $D(B_{4,4}) = 2$ whereas $\chi_D\left(B_{4,4}\right) = 4$.

Interested reader are referred to the article 'Breaking the symmetries of the book graph and the generalized Petersen graph, *SIAM J. of Disc. Math.* 23 (2009), 1200–1216' and the references therein.

3.7 Diverse areas

Representation of Cyclotomic Fields and Their Subfields. This work is on circulant and companion matrices.

Let $\mathbb{K}$ be a finite extension of a characteristic zero field $\mathbb{F}$. We say that the pair of n×n matrices (A,B) over $\mathbb{F}$ represents $\mathbb{K}$ if $\mathbb{K} \cong \mathbb{F}[A]/ < B >$ where $\mathbb{F}[A]$ denotes the smallest subalgebra of $M_n(\mathbb{F})$ containing A and $< B >$ is an ideal in $\mathbb{F}[A]$ generated by B. In particular, A is said to represent the field $\mathbb{K}$ if there exists an irreducible polynomial $q(x) \in \mathbb{F}[x]$ which divides the minimal polynomial of A and $\mathbb{K} \cong \mathbb{F}[A]/ < q(A) >$. In this work, we identify the smallest circulant matrix representation for any subfield of a cyclotomic field. Furthermore, if p is any prime and $\mathbb{K}$ is a subfield of the p-th cyclotomic field, then we obtain a zero-one circulant matrix A of size $p \times p$ such that $(A, \mathbf{J})$ represents $\mathbb{K}$, where $\mathbf{J}$ is the matrix with all entries 1. In case, the integer n has at most two distinct prime factors, we find the smallest 0-1 companion matrix that represents the n-th cyclotomic field. We also find bounds on the size of such companion matrices when n has more than two prime factors.

More on this can be found at https://arxiv.org/abs/1106.1727 and its references.

Nonsingular circulant graphs and digraphs. This work is related to polynomials divisible by cyclcotomic polynomials.

We give necessary and sufficient conditions for a few classes of known circulant graphs and/or digraphs to be singular. The above graph classes are generalized to (r, s, t)-digraphs for nonnegative integers r, s and t, and the digraph $C_n^{i,j,k,l}$, with certain restrictions. We also obtain a necessary and sufficient condition for the digraphs $C_n^{i,j,k,l}$ to be singular. Some necessary conditions are given under which the (r, s, t)-digraphs are singular.

More on this can be found at https://arxiv.org/pdf/1106.0809.pdf and its references.

Pattern polynomial graphs. This work is on adjacency algebra of a graph.

A graph X is said to be a *pattern polynomial graph* if its adjacency algebra is a coherent algebra. In this study we will find a necessary and sufficient condition for a graph to be a pattern polynomial graph. Some of the properties of the graphs which are polynomials in the pattern polynomial graph have been studied. We also identify known graph classes which are pattern polynomial graphs.

More on this can be found at https://arxiv.org/abs/1106.4745 and its references.

3.8 Linear codes and error correction

The key motivation for studying codes over $\mathbb{Z}_4$, the ring of integers modulo 4 is that they can be used to obtain desirable types of good binary codes. Such codes have been studied widely in connection with the construction of lattices, sequences with low correlation and in a variety of other contexts by many researchers.

Many good nonlinear binary codes of high minimum distances have a simple description as a linear code over $\mathbb{Z}_4$. Being a linear code decoding becomes simplified.

A linear code $\mathcal{C}$, of length n, over $\mathbb{Z}_4$ is a submodule of $\mathbb{Z}_4^n$. The minimum Hamming distance d_H of $\mathcal{C}$ is given by

$$d_H = \min\{w_H(x-y) : x, y\mathcal{C}, x \neq y\},$$

where $w_H(x)$ is the number of nonzero components in x. It is widely used for error correction/detection capabilities.

Another distance which is not that widely used is the Lee distance. Lee weight of an element $a \in \mathbb{Z}_4$, denoted $w_L(a)$ is the minimum of $\{a, 4-a\}$. Lee weight of a vector $x \in \mathbb{Z}_4^n$ is the sum of Lee weights of its components and the minimum Lee distance of $\mathcal{C}$ is

$$d_L = \min\{w_L(x-y) : x, y \in \mathcal{C}, x \neq y\}.$$

It was known that for any linear code $\mathcal{C}$ over $\mathbb{Z}_4$, d_H, the minimum Hammimg distance of $\mathcal{C}$ and d_L, the minimum Lee distance of $\mathcal{C}$ satisfy $d_H \geq \lceil \frac{d_L}{2} \rceil$. The code $\mathcal{C}$ is said to be of type $\alpha(\beta)$ if $d_H = \lceil \frac{d_L}{2} \rceil$ ($d_H > \lceil \frac{d_L}{2} \rceil$).

Professor Lal and his coauthors, defined Simplex codes of type α and β, namely, S_k^α and S_k^β, respectively, over $\mathbb{Z}_4$. Some fundamental properties like 2-dimension, Hamming and Lee weight distributions, weight hierarchy etc. were determined for these codes by them. They also showed that binary images of S_k^α and S_k^β by the Gray map give rise to some very interesting binary codes.

More on this can be found from 'On $\mathbb{Z}_4$-simplex codes and their gray images, *Applied Algebra, Algebraic Algorithms and Error-Correcting Codes.* AAECC 1999. Lecture Notes in Computer Science, vol 1719.

In another work, Professor Lal and his coauthors studied the lower bounds on the minimum number of code words of any binary code of length n such that the Hamming spheres of radius R with center at code words cover the Hamming space $\mathbb{F}_2^n$. They generalized Honkala's idea to obtain further improvements only by using some simple observations of Zhang's result. This lead to nineteen improvements of the above mentioned lower bound.

More on this can be found from 'On Lower Bounds For Covering Codes, *Designs, Codes and Cryptography*, 15, 237–243 (1998)'.

3.9 Zeor sum two person semi-Markov games

Markov (stochastic) games were introduced by Shapley (1953). A Markov game is a dynamic probabilistic game which proceeds over the infinite future (time horizon), with the property that a transition is made at every time instant. The transition may return the game to the state it previously occupied, but a transition occurs nevertheless. We want to turn our attention to a more general class of dynamic games, where the transitions may be several of the time-intervals, where this transition time can depend on the transition that is made. The game remains no longer strictly Markovian. However, the research of Professor Lal and one of his coauthors, makes it clear that it retains enough of the Markovian properties to deserve the name of a 'semi-Markov' game. They introduced and investigated the semi-Markov game and reveal the additional flexibility that brought to the problem of modelling dynamic probabilistic situations with conflict. The theory of semi-Markov games finds applications in dynamic overlapping generations models, dynamic oligopoly models, etc.

Apart from being a very fundamental piece of work that has motivated many researchers to work on this newly introduced area, this article may read for its poetic presentation and pure entertainment.

It is now a widely followed piece of work. More on this can be found in *J. Appl. Prob.* 29, 56–72 (1992).

3.10 Inequalities among two rowed immanants of the q-Laplacian of Trees and Odd height peaks in generalized Dyck paths

Let T be a tree on n vertices and let $\mathcal{L}_q^T$ be the q-analogue of its Laplacian. For a partition $\lambda \vdash n$, let the normalized immanant of $\mathcal{L}_q^T$ indexed by λ be denoted as $\overline{\mathrm{Imm}}_\lambda(\mathcal{L}_q^T)$. Schur showed that immanants of positive semidefinite matrices are known to be nonnegative. When the matrix is the Laplacian of a tree T, then, simpler (combinatorial) proofs are known. In this work, Professor Lal and his coauthors consider the q-analogue $\mathcal{L}_q^T$ of the Laplacian L of a tree T. It is known that $\mathcal{L}_q^T$ is positive semi-definite if and only if $q \in (-1, 1)$. The combinatorial proofs work even when $q \notin (-1, 1)$ and thus extend the scope of Schurs Theorem.

A string of inequalities among $\overline{\mathrm{Imm}}_\lambda(\mathcal{L}_q^T)$ is known when λ varies over hook partitions of n as the size of the first part of λ decreases. In this work, they identified a similar sequence of inequalities when λ varies over two row partitions of n as the size of the first part of λ decreases.

First, they established a very fundamental identity involving binomial coefficients and irreducible character values of $\mathfrak{S}_n$ indexed by two row partitions.

When $\lambda = n - k, k$ is a two-row partition of n, the dimension of the irreducible representation of $\mathfrak{S}_n$ is known to be related to the Catalan number and hence related to paths in the plane. Our proof can be interpreted using the combinatorics of generalized Riordan paths.

The combinatorialization of normalized immanant computations gives an expression for the normalized immanant as a sum of positive quantities where each term is split into a product of two factors, one which depends on the irreducible character (and is independent of the tree) and another which depends only on the tree (and is independent of the character values). Our main lemma is a term-by-term comparison of this expression as the partition λ varies. Our main lemma also admits a nice probabilistic interpretation involving peaks at odd heights in generalized Dyck paths or equivalently involving special descents in Standard Young Tableaux with two rows.

Two more very interesting results followed. One is when one takes the bivariate q, t-Laplacian matrix $\mathcal{L}_{q,t}^{T}$ of the tree T and for a complex number $z \in \mathbb{C}$, set $q = z$ and $t = \overline{z}$. Thus, they get a string of inequalities about the normalized two-row immanants of this asymmetric matrix.

Professor Lal and his coauthors obtained many important inequalities between $\overline{\mathrm{Imm}}_{\lambda_1}(\mathcal{L}_q^{T_1})$ and $\overline{\mathrm{Imm}}_{\lambda_2}(\mathcal{L}_q^{T_2})$ when T_1 and T_2 are comparable trees in the GTS_n poset and when λ_1 and λ_2 are both two rowed partitions of n, with λ_1 having a larger first part than λ_2.

More about this can be found at:
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