Cosmological implications of Scale Invariant GR

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Scale invariant Standard Model

We are considering standard model of particel physics, including gravitation, such that the action respects local scale invariance :

$$\mathcal{S} = \int d^4x\sqrt{-g}\left[\frac{-\beta}{4}H^\dagger H\tilde{\mathcal{R}} + g^{\mu\nu}(D_\mu H)^\dagger(D_\nu H) - \lambda(H^\dagger H)^2\right.$$

$$\left.-\frac{1}{4}g^{\mu\nu}g^{\alpha\beta}(A_{\mu\alpha}A_{\nu\beta} + B_{\mu\alpha}B_{\nu\beta} + G_{\mu\alpha}G_{\nu\beta}) - \frac{1}{4}g^{\mu\nu}g^{\alpha\beta}(E_{\mu\alpha}E_{\nu\beta})\right] + \mathcal{S}_{fermions}$$

where

$$\tilde{\mathcal{R}} = \mathcal{R} - 6f\nabla_\mu S^\mu - 6f^2 S_\mu S^\mu$$

$$D_\mu H = (\mathcal{D}_\mu - fS_\mu)H$$

$$E_{\mu\nu} = \partial_\mu S_\nu - \partial_\nu S_\mu$$

Here H is the Higgs field, $A_{\mu\nu}, B_{\mu\nu}, G_{\mu\nu}$ and $E_{\mu\nu}$ represent the field strength ternsors for the $U(1), SU(2), SU(3)$ and Weyl vector field respectively.

The scale transformations can be seen as the product of general co-ordiante transformations and the following set of transformations, called Pseudo Scale transformations,

$$x'^{\mu} = x^{\mu}$$

$$g'_{\mu\nu} = \Lambda^2 g_{\mu\nu} \;\text{ and }\; g'^{\mu\nu} = \frac{1}{\Lambda^2} g^{\mu\nu}$$

$$\Phi'(x) = \frac{1}{\Lambda}\Phi(x)$$

$$A'_{\mu} = A_{\mu}$$

$$\Psi' = \frac{1}{\Lambda^{3/2}}\Psi$$

Hence as long as general covariance is preserved, pseudo-scale invariance is equivalent to local scale invariance.

Localization of this scale transformation introduces a vector field, S_μ .

Correspondingly, we covariantly change the derivatives of the fields and metric due to this local symmetry as follows:

$$\partial_\mu \Phi \longrightarrow (\partial_\mu - f S_\mu)\Phi \ ,$$

$$\partial_\mu g_{\alpha\beta} \longrightarrow (\partial_\mu + 2f S_\mu) g_{\alpha\beta} \ \text{and} \ \partial_\mu g^{\alpha\beta} \longrightarrow (\partial_\mu - 2f S_\mu) g^{\alpha\beta}$$

$$\partial_\mu \Psi \longrightarrow (\partial_\mu - \frac{3}{2} f S_\mu)\Psi$$

And under scale transformation,

$$S_\mu \longrightarrow S_\mu - \frac{1}{f}\partial_\mu \ln(\Lambda)$$

The transformation properties of fermion fields are deduced from the Dirac lagrangian in curved space time.

It can be shown that the Dirac lagrangian remains the same even after these substitutions. So, S_μ does not interact with any of the leptons or quarks, etc..

The model

Let us consider the scale invariant lagrangian for a real scalar field as follows :

$$\mathcal{L} = -\frac{\beta}{8}\Phi^2\tilde{\mathcal{R}} \; + \; \frac{1}{2}g^{\mu\nu}\mathcal{D}_\mu\Phi\mathcal{D}_\nu\Phi - \frac{\lambda}{4}\Phi^4 - \frac{1}{4}g^{\mu\rho}g^{\nu\sigma}E_{\mu\nu}E_{\rho\sigma}$$

where, $\mathcal{D}_\mu\Phi = (\partial_\mu - fS_\mu)\Phi$ and $E_{\mu\nu} = \partial_\mu S_\nu - \partial_\nu S_\mu$.

Here we considered a real scalar field, instead of Higgs field, as a prototype, to study the cosmological implications of scale invariant gravitational action.

Field equations and EOMs

Now, taking the variation of the above action gives us the modified Einstein's field equations and the equations of motion of the scalar and vector fields respectively as follows.

Modified Einstein's field equations :

$$\Phi^2 B_{\mu\nu} + \partial_\lambda(\Phi^2) C^\lambda{}_{\mu\nu} + (\Phi^2)_{;\rho;\sigma} D_{\mu\nu}{}^{\rho\sigma} = \frac{4}{\beta} T_{\mu\nu}$$

where,

$$\begin{aligned}
B_{\mu\nu} &= \mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R} g_{\mu\nu} + 3f^2 g_{\mu\nu} S\cdot S - 6f^2 S_\mu S_\nu \,, \\
C^\lambda{}_{\mu\nu} &= -3f S^\lambda g_{\mu\nu} + 3f(g^\lambda_\mu S_\nu + g^\lambda_\nu S_\mu)\,, \\
D_{\mu\nu}{}^{\rho\sigma} &= -\frac{1}{2}(g^\rho_\mu g^\sigma_\nu + g^\rho_\nu g^\sigma_\mu) + g_{\mu\nu} g^{\rho\sigma} \,, \\
T_{\mu\nu} &= -\mathcal{L}_{SV} g_{\mu\nu} + \mathcal{D}'_\mu \Phi \mathcal{D}'\nu \Phi - E_{\mu\rho} E_{\nu\sigma} g^{\rho\sigma} \,, \\
\text{and } \mathcal{L}_{SV} &= \frac{1}{2} g^{\mu\nu} \mathcal{D}'_\mu \Phi \mathcal{D}'_\nu \Phi - \frac{\lambda}{4}\Phi^4 - \frac{1}{4} g^{\mu\rho} g^{\nu\sigma} E_{\mu\nu} E_{\rho\sigma} \,.
\end{aligned}$$

Equation of motion for the scalar field :

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\Phi - f\Phi\nabla_\mu S^\mu - f^2 S_\mu S^\mu\Phi + \lambda\Phi^3 + \frac{\beta}{4}\Phi\tilde{\mathcal{R}} = 0$$

Equation of motion for the vector field :

$$\nabla_\mu E^{\mu\nu} = \left(1 + \frac{3\beta}{2}\right) f\Phi\mathcal{D}'^{\nu}\Phi$$

The symmetry is broken through cosmological symmetry breaking. Here we assume that, at leading order, all the fields are independent of space coordinates and depend only on time. In this case the equations simplify considerably for,

$$\Phi(x) = \eta(t) + \xi(x)$$
$$S_\mu(x) = (S_0(t) + C_0(x), S_i(t) + C_i(x))$$

The equation of motion for S_0 turns out to be $fS_0 = \frac{\dot{\eta}}{\eta}$. Thus we choose $S_0 = 0$ as a gauge choice, which makes $\eta = constant$.

The resulting 0-0 and i-i components of the Einstein's equations can then be written as,

$$3\eta^2\frac{\dot{a}^2}{a^2} = \frac{4}{\beta}\left[\frac{\lambda}{4}\eta^4 + \frac{\dot{S}_i^2}{2a^2} + \left(1+\frac{3\beta}{2}\right)\frac{f^2\eta^2S_i^2}{2a^2}\right]$$

and

$$3\eta^2\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = \frac{4}{\beta}\left[\frac{3\lambda}{4}\eta^4 - \frac{\dot{S}_i^2}{2a^2} + \left(1+\frac{3\beta}{2}\right)\frac{f^2\eta^2S_i^2}{2a^2}\right]$$

respectively. And the equations for η and S_i become,

$$3\eta^2\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = \frac{4}{\beta}\left[\frac{\lambda}{2}\eta^2 + \left(1+\frac{3\beta}{2}\right)\frac{f^2S_i^2\eta^2}{2a^2}\right] ,$$

$$\ddot{S}_i + \frac{\dot{a}}{a}\dot{S}_i + \left(1+\frac{3\beta}{2}\right)f^2\eta^2S_i = 0 .$$

Cosmological solution

Now, we try to solve the field equations and eqautions of motion mentioned above.

The equation of motion for S_i, is similar to that of a damped harmonic oscillator, with weakly time-dependent frequency and decay terms. We seek a solution of the form

$$S_i = n_i \mathcal{S}$$

where n_i is a constant unit vector. The solution for $\mathcal{S}$ can be expressed as,

$$\mathcal{S} \;=\; Re\left\{Ae^{-\int \frac{\dot{a}}{2a}dt - i\int \omega_1 dt} + Be^{-\int \frac{\dot{a}}{2a}dt + i\int \omega_1 dt}\right\},$$

where, A and B are assumed to be slowly varying functions of time and $\omega_1^2 = \omega^2 - \frac{H^2}{4}$, $\omega^2 = \left(1 + \frac{3\beta}{2}\right) f^2\eta^2 = M_S^2$ and $H = \dot{a}/a$.

By substituting this solution in EOM of S_i and neglecting second derivatives of A and B, we get,

$$\frac{\dot{A}}{A} + \frac{\dot{\omega}_1}{2\,\omega_1} = i\frac{\dot{H}}{4\omega_1} \quad \Rightarrow \quad A = \frac{k_1}{\sqrt{\omega_1}} e^{\frac{i}{2} sin^{-1}\frac{H}{2\omega}}$$
$$\frac{\dot{B}}{B} + \frac{\dot{\omega}_1}{2\,\omega_1} = -i\frac{\dot{H}}{4\omega_1} \quad \Rightarrow \quad B = \frac{k_2}{\sqrt{\omega_1}} e^{-\frac{i}{2} sin^{-1}\frac{H}{2\omega}}$$

where, k_1 and k_2 are constants of integration and are, in general, complex. Since $\omega >> H$ we see that A and B vary very slowly with time compared to other terms in S_i. The most rapidly varying terms are those containing $\int \omega_1 dt$ in the exponent. Due to these terms, S_i fluctuates rapidly with time.

Following this, we obtain a leading order solution and a correction to this leading order solution.

At leading order we can assume that A and B are time independent. The leading order solution for $\mathcal{S}$ can then be written as

$$\mathcal{S} = \frac{1}{\sqrt{a}}(A' \cos\theta + B' \sin\theta)$$

where $\theta = \int \omega_1 dt$ and A' & B' are some real constants. We define,

$$\begin{aligned} \rho_{S_i} &= \frac{1}{2a^2}\dot{S}_i^2 + \frac{1}{2a^2}\omega^2 S_i^2 \\ \text{and } -3P_{S_i} &= -\frac{1}{2a^2}\dot{S}_i^2 + \frac{1}{2a^2}\omega^2 S_i^2 \end{aligned}$$

as the energy density and pressure terms corresponfing to vector field in modified Einsteins' equations. We find that, at leading order,

$$\begin{aligned} \rho_{S_i} &= \frac{1}{2a^3}\left(A'^2 + B'^2\right)\omega^2 \ , \\ P_{S_i} &= 0 \ . \end{aligned}$$

We next calculate the corrections to the leading order result by taking into account the time dependence of the coefficients A and B. Now we have,

$$\begin{aligned} \mathcal{S} &= \frac{1}{\sqrt{\omega_1 a}}\left[Q\cos(\theta - x) + P\sin(\theta - x)\right] = \frac{1}{\sqrt{\omega_1 a}}U \\ \dot{\mathcal{S}} &= \frac{1}{\sqrt{\omega_1 a}}\left[\frac{H}{2}\left(\frac{\dot{x}}{\omega_1} - 1\right)U + (\omega_1 - \dot{x})V\right] \end{aligned}$$

where,

$$\begin{aligned} U &= N\cos\theta + M\sin\theta \ , V = -N\sin\theta + M\cos\theta \\ \text{and } M &= P\cos x + Q\sin x \ , N = -P\sin x + Q\cos x\,. \end{aligned}$$

Here, $x = \frac{1}{2}\sin^{-1}\frac{H}{2\omega} = \frac{1}{2}\cos^{-1}\frac{\omega_1}{\omega}$, $\dot{x} = \dot{H}/4\omega_1 = -\dot{\omega}_1/H$ and P & Q are some real constants.

With this substitution in the field equations, we get,

$$\rho_{S_i} = \frac{(P^2+Q^2)M_S}{2a^3} + \frac{(P^2+Q^2)\lambda\eta^2}{48\beta a^3 M_S} + \frac{(P^2+Q^2)(A'^2+B'^2)M_S}{6\beta\eta^2 a^6},$$

$$P_{S_i} = \frac{(P^2+Q^2)(A'^2+B'^2)M_S}{24\beta\eta^2 a^6}.$$

The leading term varies as a^{-3} as already found earlier. Here, we also find two subleading terms. One of these falls as $1/a^3$, also suppressed by a factor of $'\lambda'$ and the second term, falls much faster, as a^{-6}, as the universe expands. We also get a small correction term to P_{S_i}, which also decays rapidly as a^{-6}, as the universe expands.

The 0-0 component of the Einstein's equations can, now, be written as,

$$\frac{3\beta}{4}\eta^2 H^2 = \frac{\lambda}{4}\eta^4 + \frac{(P^2+Q^2)}{2\omega_1 a^3}\left(\omega^2 - \frac{\dot{H}}{4}\right) .$$

which can be cast in the form,

$$1 = \Omega_\Lambda + \Omega_{S_i}$$

where

$$\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_{cr}} \, , \Omega_{S_i} = \frac{\rho_{S_i}}{\rho_{cr}} \, ,$$

$$\rho_\Lambda = \frac{\lambda}{4}\eta^4 \quad , \rho_{S_i} = \frac{(P^2+Q^2)}{2\omega_1 a^3}\left(\omega^2 - \frac{\dot{H}}{4}\right) \text{ and } \rho_{cr} = \frac{3\beta}{4}\eta^2 H^2 \, .$$

This expression looks like the ΛCDM model with $\Omega_M = \Omega_{S_i}$.

We also find that, $P^2 + Q^2 \approx \frac{3M_P^2 H_0^2 \Omega_{M0}}{4\pi M_S}$.

Cosmological evolution of different components including radiation

Here, we setup a set differential equations to study the dynamical evolution of different components of the universe, including radiation.

For this study, it is convenient to introduce the folllowing dimensionles variables:

$$X^2 = \frac{\lambda}{3\beta}\frac{\eta^2}{H^2} = \Omega_\Lambda\,,$$

$$Y^2 = \frac{2}{3\beta}\frac{\dot{\mathcal{S}}^2}{a^2\eta^2} = \Omega_1\,,$$

$$Z^2 = \frac{2}{3\beta}\left(1+\frac{3\beta}{2}\right)\frac{f^2\mathcal{S}^2}{a^2H^2} = \Omega_2\,,$$

$$R = \frac{4}{3\beta}\frac{\rho_{R,0}}{a^4\eta^2H^2} = \Omega_R\,.$$

Here, $\Omega_{S_i} = \Omega_1 + \Omega_2$ and the prime denotes derivative with respect to $\ln a$.

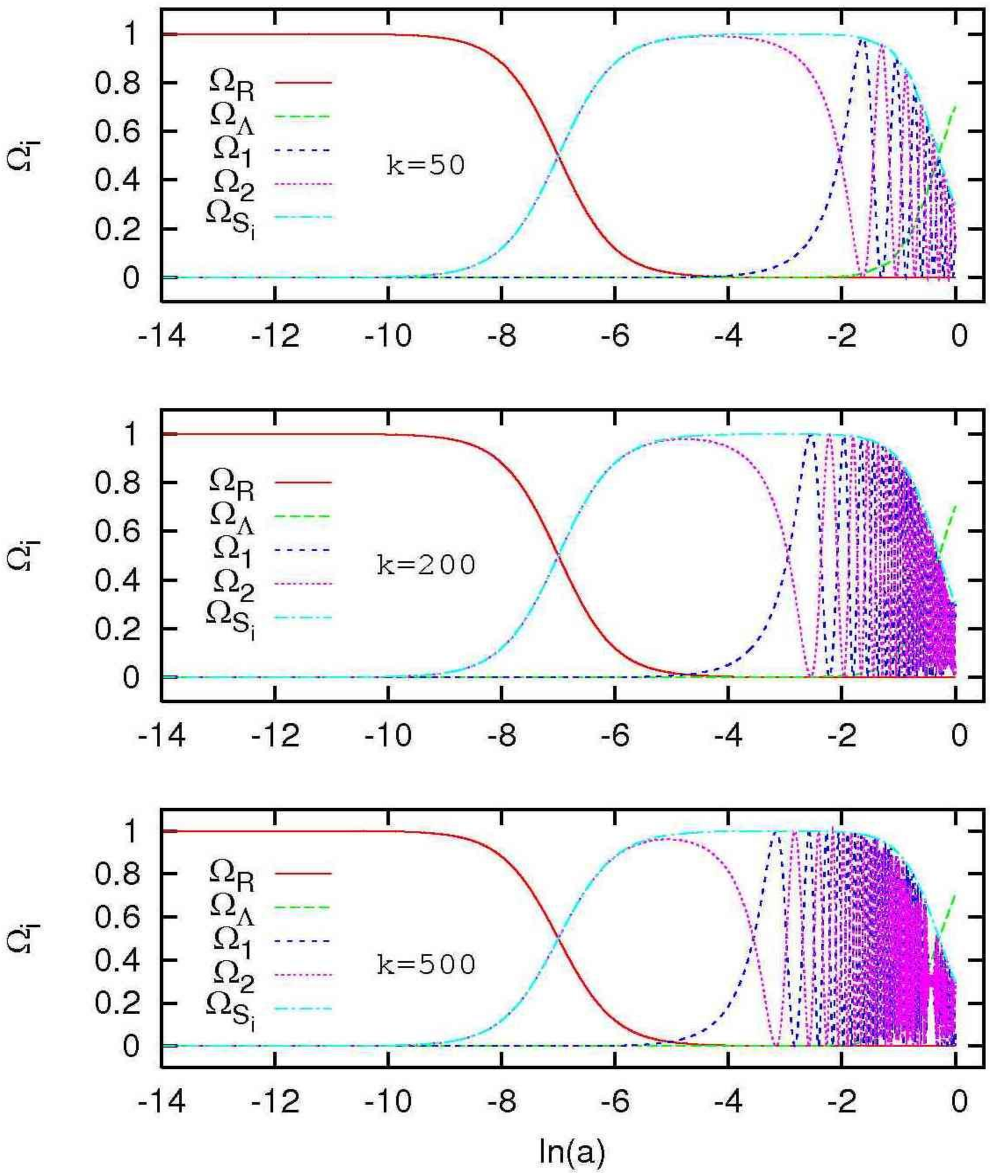
Thus, the modified Einstein's field equations and the equations of motion of the scalar and vector fields take the form of the following set of differential equations :

$$\begin{aligned} X' &= X(2 - 2X^2 - Z^2)\,, \\ Y' &= -Y(2X^2 + Z^2) - \frac{\kappa}{2}XZ\,, \\ Z' &= Z(1 - 2X^2 - Z^2) + \frac{\kappa}{2}XY\,, \\ R' &= -2R(2X^2 + Z^2)\,, \end{aligned}$$

where, $\kappa = \sqrt{\frac{12\beta}{\lambda}}\frac{\omega}{\eta} = \sqrt{3}M_P M_S/\sqrt{2\pi\rho_V}$ and $'$ denotes derivative with respect to $\ln a$. So,

$$f' = \frac{df}{d\ln a}$$

The typical evolution of these variables from the beginning of radiation dominated era($\ln a = -29$) to current era($\ln a = 0$) have been studied and presented in the following graph.



As seen here, the vector field undergoes rapid oscillations depending on its mass: higher the mass(k), higher the frequency of oscillations and vise versa.

Conclusions

- In the present work, a locally scale invariant generalization of GR has been analyzed.

- The symmetry is broken by a recently introduced mechanism called, cosmological symmetry breaking.

- The equations of motion lead to a small, but non-zero, cosmological constant or Dark energy.

- The cold dark matter arises in the form of vacuum oscillations of the Weyl's gauge field.

- A numerical study has been carried out and the standard LambdaCDM model paradigm has been reproduced.

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Thank You

The prime references for this work are :

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