Department of Physics, IIT-Kanpur

PH D admission test 20.05.2014 Time : 2 hrs. Total Marks: 100

Please answer all the questions serially. Otherwise they will not be graded.

Q1a) Consider an electron with charge $q = -e$ at rest in presence of a magnetic field $\mathbf{B} = B\hat{z}$.
i) Write down the Hamiltonian in matrix form (neglect the orbital motion). Obtain the energy eigenvalues and the corresponding spinors. **[3]**
ii) Write down the time-evolution operator for this system. Assuming that at $t = 0$ the spinor is given by

$$\xi(t=0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Find the spinor at a later time t i.e. $\xi(t)$. **[4]**
iii) What is the probability of finding the electron with $\langle S_x \rangle = -\hbar/2$ at a later time t? **[3]**
iv) Calculate the average value of S_z at a later time t. **[2]**

Q1b) Consider a simple harmonic oscillator whose quantum mechanical Hamiltonian is given by

$$H = \frac{\epsilon}{2}\Big[-\frac{d^2}{d\eta^2} + \eta^2\Big],$$

where ϵ is the energy unit and η is the dimensionless position coordinate. Its orthonormal eigenket is represented by $|n\rangle$ and the raising and lowering operators are, respectively,

$$b^\dagger = \frac{1}{\sqrt{2}}\Big[\eta - \frac{d}{d\eta}\Big], \qquad b = \frac{1}{\sqrt{2}}\Big[\eta + \frac{d}{d\eta}\Big].$$

i) Using $|n\rangle, b, b^\dagger$ and their various properties, calculate $\langle n'|\eta|n\rangle$. **[2]**
ii) Using the previous result, write down the matrix representation of the position operator η. **[3]**
iii) Calculate $\langle n|\frac{d^2}{d\eta^2}|n\rangle$. **[3]**

Q2a) Consider an invertible matrix A

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

Find the eigenvalues and the corresponding normalized eigenvectors.

Now consider another matrix B

$$B = ADA^{-1}$$

where D is a diagonal matrix

$$D = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

Find the eigenvalues and eigenvectors of B. **[8 + 2]**

Q2b)
i) Show that in an arbitrary central force field the orbit of a particle will be in a two dimensional plane. **[2]**
ii) Consider a central force field where the potential is given as

$$V(r) = -\frac{A}{r^n}$$

where A is some arbitrary positive constant.
Write down the one-dimensional effective potential in radial direction.
Show that for $(n < 2)$, the system admits a stable circular orbit.
Comment on the existence and the stability of circular orbits for $(n \geq 2)$ **[2 + 3 + 3]**

Q3a) A particle of mass M and charge Q starts from the origin (x=y=z=0) with a speed V. The velocity has equal components in all the three directions. There is a uniform magentic field $\vec{B} = B_0\hat{z}$, and an acceleration due to gravity along $-\hat{z}$ direction.
(i) Plot the displacement along z direction as a function of time.
(ii) Sketch the trajectory of the particle.
(iii) How many rotations the particle would make by the time its displacement along z axis becomes zero.
(iv) Estimate the trajectory length covered by the particle by that time. **[3 + 3 + 3 + 3]**

Q3b) Consider an electric dipole $\vec{p} = p_0\hat{z}$ located at $\vec{r}_0 \equiv (0, 0, 1)$, in an external potential $V_{ext}(\vec{r}) = a_0 r^2$. Find the energy of the dipole, and the force on the dipole. **[4 + 4]**

Q4a) Consider N noninteracting particles moving in a large three-dimensional enclosure of volume V. The energy of a particle depends on the magnitude of its momentum, $\epsilon(\vec{p}) = v_0|\vec{p}|$. The enclosure is in thermal equilibrium through a thermal contact with a reservoir at temperature T. Find (i) the equation of the state (ii) the internal energy and (iii) the specific heat of the system.
[5 + 4 + 1]

Q4b) Consider a plane monochromatic optical beam, linearly polarized, incident on a Youngs double slit. The distance between the two slits is d. The width of the slit is negligible compared to d and the wavelength of the optical field, λ.
i) Choosing the middle of the slits as the origin of your co-ordinate system, estimate the location of the second principal maxima on a screen at a distance L from the slits. **[2]**
ii) If the incident intensity of the optical field is I_0, what is the intensity at the first principal maxima? **[2]**
iii) Draw the resulting intensity pattern as a function of the position on the screen. **[2]**
iv) Now suppose one of the slits is blocked with a quarter wave plate, with the axis oriented along the direction of the incident polarization. This generates circular polarization. Draw the resulting intensity pattern, as observed on the screen. **[4]**

Q5a) A square pulse of height 5V and pulse width of 1ms is given as input to the following two circuits. For each case, give the circuit, write down the expression for the output voltage, and give a qualitative sketch of the output waveform as would be seen on an oscilloscope :
i) A 0.1H inductor connected in series to 10 Ω resistor which is grounded on the other side. The input voltage is given across ground and the inductor, and the output voltage is measured across the resistor.
ii) A differentiator using an ideal operational amplifier, with values of your choice for the components you use. Justify the values of components. **[5 + 5]**