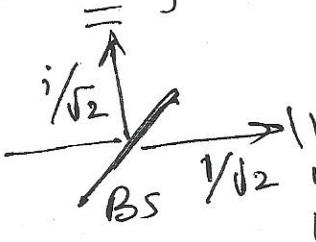
The Interferometer detector.

Let us take a look at the Michelson Interferometer as the simplest g.w. detector and see how to detect the g.w. phase shift.



$L+\delta l = L+\frac{hL}{2}$

Phase change

$$\delta\phi = 2\times k\cdot 2\delta l$$
$$= 4\cdot k\frac{hL}{2}$$
$$= \frac{4\pi L h}{\lambda}$$

In more detail:

The field at the output port (asymm. port)

$$E_o = \frac{E_{in}}{2}\left[e^{i2kL_x} + e^{i2kL_y}\right]$$

$$P_{out} = \frac{P_{in}}{4}\left[1+1+e^{i2k(L_x-L_y)} + e^{-i2k(L_x-L_y)}\right] = \frac{P_{in}}{2}\left(1+\cos 2k\Delta L\right)$$
$$= \frac{P_{in}}{4}\left[2+2\cos 2k(L_x-L_y)\right]$$
$$\rightarrow P_{in}\cos^2 k(L_x-L_y)$$

At a dark fringe, $P_{out} = P_{in}\sin^2 k(L_x-L_y)$
$$= P_{in}\sin^2 k\left(\Delta L_{fixed} + 2\delta L(t)\right)$$
$$= P_{in}\sin^2 k\left(\Delta L_{fix} + Lh\right)$$
$$\simeq P_{in}\cdot\sin^2 k\delta l \simeq P_{in}\left(\frac{2\pi\delta L}{\lambda}\right)^2$$

$\propto h^2$ Too small!

So, trying to get an output signal that is linear in δl or h is a fundamental problem to be solved.

Imp: The response goes to zero at some ω_g!

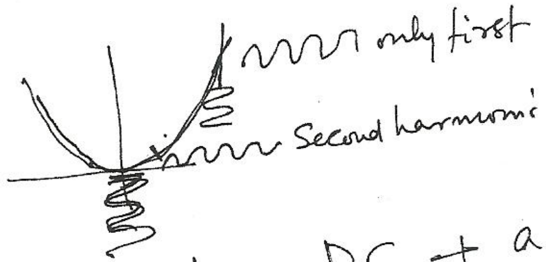
We need a Signal that is proportional to slope of the intensity with δL

$$\frac{dI}{d\phi} \propto \phi \propto h$$

Need a slope detector.

$I \propto \phi^2$

$\frac{dI}{d\phi} \propto \phi$.



So output = DC + $a \sin \omega t + b \sin 2\omega t + \ldots$

The amplitude $a \propto$ slope $\underline{\propto h}$

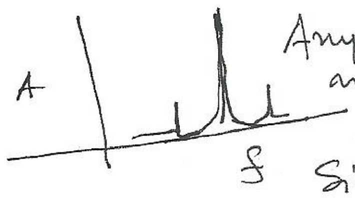
To get this out, we multiply by $\sin \omega t$

and get DC. $\sin \omega t + a \sin^2 \omega t + b \sin \omega t \sin 2\omega t$

Average with a low pass filter $\rightarrow \underline{\frac{a}{2}}$!

There is another way to look at this problem.

In frequency space



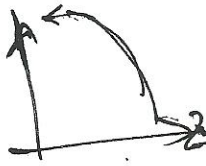
Any harmonic modulation will create side bands.

We can allow carrier to leak through and we will directly get the beats between carrier and side bands, but nonlinear (DC readout) $\rightarrow$ low noise laser necessary.

We can take to a reference beam from the input laser and mix it with light sidebands at output port $\rightarrow$ homodyne.

We can also modulate the light at input and measure the beat between created sideband and the small amount of carrier leaking out ... this is GW signal $\rightarrow$ heterodyne

One thing to note is that if $\Delta L = 0$, light at ~~&~~ all ~~wa~~ frequencies produce a dark fringe at output, ~~if one is at~~ That is, ~~Created~~ while the phase is adjusted to get a dark fringe at output (asymmetric port), created carriers from modulation also produce a dark fringe. So if we need, as it does in GW detectors, ~~carriers~~ side bands to leak out to get the without distructive interference $\Delta L \neq 0$. This is called the Schnupp asymmetry.



Rotating frame

$$\cancel{\frac{2\pi}{\lambda_m}(L_x - L_y) = \pi/2}$$

$$\cancel{\frac{\lambda}{2\pi}_{\omega+\Omega} = \frac{c}{\omega+\Omega} = \frac{c}{\omega(1+\Omega/\omega)}}$$

$$\frac{\lambda_{\nu_0+\nu}}{2\pi} = \frac{c}{\nu_0+\nu} = \frac{c}{\nu_0(1+\nu/\nu_0)} \approx \lambda_{\nu_0}(1+\nu/\nu_0)$$

$$L_x - L_y \simeq \lambda_{\nu_0}(1+\nu/\nu_0)/2$$

$$L_x - L_y = n\lambda_{\nu_0} \quad \frac{\Delta L}{\lambda}\frac{\nu}{c} \approx \pi/4$$

$$\lambda_{mod}/8$$

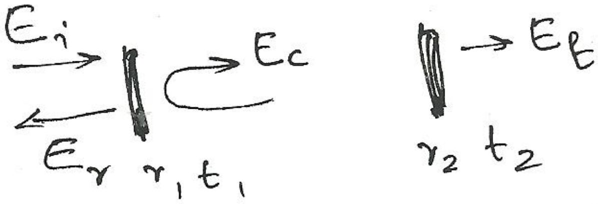
$$2 \times \frac{2\pi}{\lambda_{mod}}(L_x - L_y) = \pi/2$$

$$L_x - L_y = \frac{\lambda_{mod}}{8}$$

$$30\,\text{MHz} \sim \left(\frac{3\times10^8}{3\times10^7}\right)/8 \quad 30\,\text{MHz} \rightarrow 1.25\,\text{m}$$

This can be (is) used for locking the M-IFO on the dark fringe. Any carriers leaking out will beat with the carrier and give a beat signal ~~driving that~~ linear in both → driving this back to zero keeps the interferometer on the dark fringe.

Now we take a look at the real signal amplifier – the Fabry-Perot Cavity.



$$E_t = E_i \left[t_1 t_2 e^{-ikL} + r_2 r_1 e^{-i2kL} t_1 t_2 e^{-ikL} + \cdots \right.$$

$$= \frac{t_1 t_2 e^{-ikL}}{1 - r_1 r_2 e^{-i2kL}} E_i$$

$$E_r = E_i \left[-r_1 + t_1^2 r_2 e^{-i2kL} + t_1^2 r_2 r_1 r_2 e^{-i4kL} + \cdots \right]$$

$$= E_i \left[-r_1 + t_1^2 r_2 \left(\sum_0^\infty r_1 r_2 e^{-i2kLn} \right) \right]$$

$$= \left[-r_1 + \frac{t_1^2 r_2 e^{-2ikL}}{1 - r_1 r_2 e^{-2ikL}} \right] E_i$$

$$E_c = t_1 E_i + r_1 r_2 e^{-i2kL} E_c$$

$$\rightarrow \frac{t_1 E_i}{1 - r_1 r_2 e^{-i2kL}}$$

$r_1^2 + t_1^2 = 1$

$r_1^2 + t_1^2 + L = 1$
when there is loss.

When $2kL = n(2\pi)$, $L = n\lambda/2$ Resonance.

And $1 - r_1 r_2 \simeq 0 \longrightarrow$

The reflected field is very important in g.w. interferometer physics.

$$E_r = E_i \left(-r_1 + \frac{t_1^2 r_2 e^{-i2kL}}{1 - r_1 r_2 e^{-i2kL}}\right) \rightarrow R$$

$$= E_i \left[\frac{-r_1 (1 - r_1 r_2 e^{-i2kL}) + t_1^2 r_2 e^{-i2kL}}{(R)}\right]$$

$$= E_i \left[\frac{-r_1 + r_1^2 r_2 e^{-i2kL} + t_1^2 r_2 e^{-i2kL}}{(R)}\right]$$

$$= E_i \left[\frac{-r_1 + r_2 (r_1^2 + t_1^2) e^{-i2kL}}{(R)}\right]$$

$$= E_i \left[\frac{-r_1 + r_2 (1 - L_i) e^{-i2kL}}{1 - r_1 r_2 e^{-i2kL}}\right]$$

At resonance $e^{-i2kL} = 1$.

1. $E_r \rightarrow 0$!
2. $|r_2| > |r_1|$ Over coupled.
3. ~~far~~ far from resonance $\frac{E_r}{E_i} = -1$
4. At resonance and close to it with $r_2 \simeq 1$

$$E_r \simeq E_i \left(-r_1 + \frac{t_1^2 r_2 e^{-2i\delta}}{1 - r_1 r_2 e^{-2i\delta}}\right)$$

$$\approx E_i \left(-r_1 + \frac{(1 - r_1^2)\, 1}{(1 - r_1)(1 - i2\delta)}\right)$$

E_r does not change much.

$$\approx E_i \left[-r_1 + \frac{(1 - r_1^2)}{(1 - r_1)(1 - r_1 i2\delta)}\right]$$ small.

$$\approx E_i$$

To get familiar with Cavity optics let us look at a situation with $t_1 = t_2$, $r_1 \approx r_2$ etc. $E_t = \frac{E_0 t^2}{1 - r^2 e^{-i\Delta}}$

$$I_{out} = \frac{I_0 T^2}{|1 - Re^{-i\Delta}|^2}$$

$$|1 - Re^{-i\Delta}|^2 = (1 - Re^{-i\Delta})(1 - Re^{+i\Delta})$$

$$= 1 + R^2 - 2R\cos\Delta$$

$$= (1 + R^2 - 2R) + 2R - 2R\cos\Delta)$$

$$= (1-R)^2 + 2R(1 - \cos\Delta)$$

$$= (1-R)^2 + 4R\sin^2\Delta/2)$$

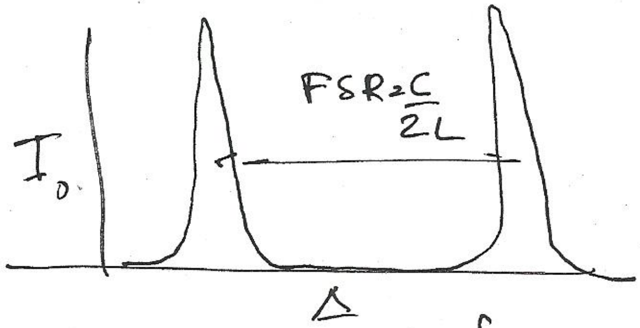
$$I_T = \frac{I_0 T^2}{(1-R)^2\left[1 + \frac{4R}{(1-R)^2}\sin^2\frac{\Delta}{2}\right]}$$

$$= \frac{I_0 T^2}{(1-R)^2\left[1 + \underbrace{F}_{\text{Finesse Function}}\sin^2\frac{\Delta}{2}\right]}$$

At resonance $\sin\frac{\Delta}{2} = 0$, $I_T = \frac{I_0 T^2}{(1-R)^2} = I_0$

All pass!

Even if both mirrors are $\approx 100\%$ reflective, nothing gets reflected!



When $I_0\left(1+F\sin^2\frac{\Delta}{2}\right)^{-1} = I_0/2$

$$\frac{1}{1+F\sin^2\frac{\Delta}{2}} = \frac{1}{2}$$

$$F\sin^2\frac{\Delta}{2} = 1, \Rightarrow F\frac{\Delta^2}{2} = 1 \rightarrow \Delta = \sqrt{\frac{2}{F}}$$

$$= 2\sqrt{\frac{2}{4R/(1-R)^2}} = \frac{\sqrt{2}(1-R)}{2\sqrt{R}}$$

$$= \frac{\cancel{\sqrt{2}}(1-R)}{\sqrt{2R}}$$

What is called finesse of the cavity is

$$\frac{FSR}{\Delta\omega} = \frac{c/2L}{\Delta\omega} = \frac{c/2L}{(\Delta)_{1/2}\cdot 1/c} = \frac{\pi\sqrt{R}}{1-R}$$

So 99% Reflecting means. $F \simeq \frac{\pi\cdot(1-.005)}{\cancel{0}.01}$
$\simeq 300$

etc.

With different R and T

$$I_{out} = \frac{I_0 T_1 T_2}{\left(1-\sqrt{R_1}\sqrt{R_2}\right)^2}$$

$1-.9999$
$.0001+$

Important.

~~T_1 98%. T_2 99.99% etc.~~ $R_1 \sim 98\%$.
$R_2 \sim 99.99\%$.

T_{1}, T_{th} I_{in} $\cancel{= \frac{I_0\cdot(0.98)\times 1}{(1-(.01))^2}}$ $= I_{in} \simeq \frac{I_0\times(.02)\cdot .00}{(1-\sqrt{.98})^2}$

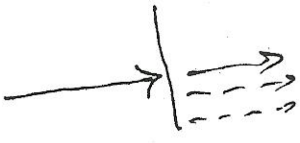
$(.01)^2$
$1-\cancel{(1-.01)}$

~~Aoph~~ Useful

We see that the useful workings of an F-P cavity is all in the domain of *phase* of the optical field and not in intensity. Therefore, we need to know how to measure light fields with their phase information intact. Of course, only *relative phases matter*.

① The reflected field is phase flipped by π. That is just mirror reflection. $(-r_1)$.

Relative to this, the cavity field that may leak out (finite r_1 and t_1) is in phase with the input field. So the reflected and leaked fields cancel and hence no reflected field.



So, a reflected field away from ω_0 does not get affected by anything happening inside the cavity. They can be used as stable references for phase detection! That is the basic idea for Pound-Drever-Hall detection of phase, locking etc.

From the beam that goes in we can generate a phase coherent beam of a different frequency by phase modulation.

$$E = E_0 \, e^{i(\omega t + b \sin \Omega t)}$$

$\hookrightarrow$ phase $\phi \to \phi(t)$ (not frequency)

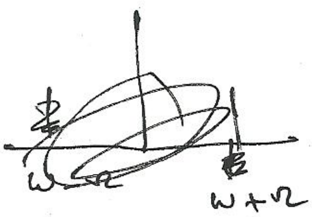
When b is small

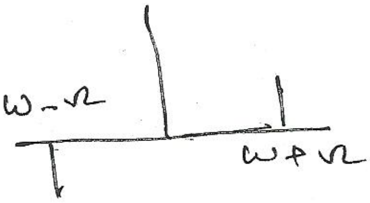
$$E = E_0 e^{i\omega t} e^{ib \sin \Omega t}$$

$$\approx E_0 e^{i\omega t}\left[1 + ib \sin \Omega t\right]$$

$$= E_0 e^{i\omega t}\left[1 + \frac{b}{2}\left(e^{i\Omega t} - e^{-i\Omega t}\right)\right]$$

Side bands at $\omega + \Omega$ and $\omega - \Omega$, out of phase



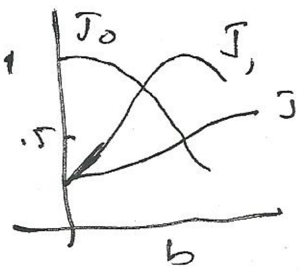


A more formal and correct expression is

$$\cancel{E_0 e} \; E_0 e^{i(\omega t + b \sin \Omega t)}$$

$$= E_0 e^{i\omega t}\left[J_0(b) + 2i J_1(b) \sin \Omega t\right]$$

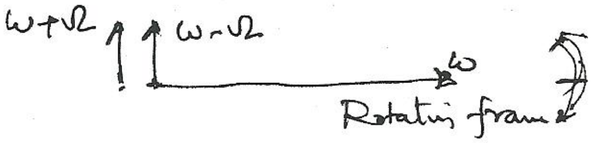
$$= E_0 \left[J_0(b) e^{i\omega t} + J_1(b) e^{i(\omega+\Omega)t} - J_1(b) e^{-i(\omega-\Omega)t}\right]$$



For small x, $J_0(x) = 1 - \left(\frac{x}{2}\right)^2$

$J_1(x) = x/2$ etc

When cavity is in resonance.



No amplitude modulation

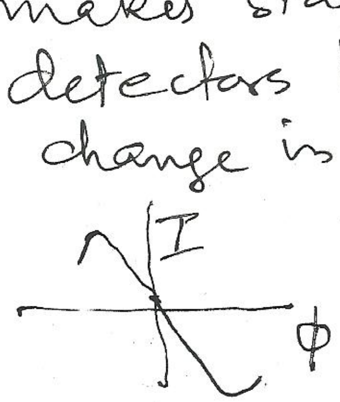
When cavity is slightly ~~out~~ out of resonance, the field has a small imaginary component

$$r_c = -r + \frac{r e^{i\delta} t^2}{1 - r^2 e^{i\delta}} \rightarrow (1 + i\delta) \quad \text{etc.}$$

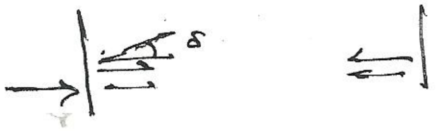
Asymmetry proportional to δ

$E_c \delta$

So, in this case we have manged to get a signal proportional to δ, by beating with $(\omega + \Omega)$ and $(\omega - \Omega)$ fields. This is essentially the Pound-Drever-Hall scheme. This is at present the most important technical input that makes stable and sensitive GW detectors possible, with ~~shift of~~ fringe change in detected light intensity proportional to the strain, while operating on the dark fringe



The field out of cavity close to resonance



$i E_c (\delta\phi)$

$= i \sqrt{P_c}(\delta\phi)$ → $P_{carrier}$

The sidebands that act as reference is $i 2\sqrt{P_s} \sin \Omega t$ → $P_{sideband}$

Total field is $E = E_{co} + E_s$

$= E_c(\delta\phi) + 2E_s \sin\Omega t$

$P_{out} = P_c(\delta\phi)^2 + 4P_s \sin^2\Omega t$

$+ 4\sqrt{P_cP_s}\,\delta\phi \sin\Omega t$

This is an amplitude modulated signal at frequency Ω and amplitude $4\sqrt{P_cP_s}\,\delta\phi$, linear in h because $\delta\phi \sim h/\lambda_e$

(actually $\delta\phi \sim \frac{2\pi h}{\lambda}\cdot F$)

'Demodulating' Pout by multiplying with $\sin\Omega t$ and averaging (low pass)

$P_{out} = \langle 4\sqrt{P_cP_s}\,\delta\phi(t) \sin\Omega t \cdot \sin\Omega t \rangle$

$= 2\sqrt{P_cP_s}\,\delta\phi$

$\delta\phi = \frac{2\pi}{\lambda}(\delta l)\cdot F$

$\approx hFL/\lambda$

Voltage on a photodiode

$V_{out} = \eta_{ph}\, 2\sqrt{P_{in}}\, J_0(b)\sqrt{P_{in}}\, J_1(b) \frac{Fl}{\lambda} h$

efficiency

$= \eta_{ph}\, 2\, P_L\, J_0(b) J_1(b) \sqrt{F_{PR}}\, F_c \frac{L}{\lambda} h$

$P_{in} = P_{Laser} \times F_{PR}$

Power Recycling Finesse