Optics of GW detectors

Review of optics

Pradeep Kumar K

Center for Lasers & Photonics
Indian Institute of Technology Kanpur

Outline

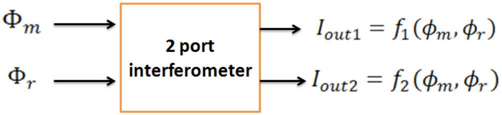
1. Electromagnetic fundamentals
2. Describing optical elements
3. Michelson interferometer
4. Fabry-Perot cavity
5. Higher-order transverse modes

Detecting optical signals

- Sinusoidal optical signals characterized by amplitude/power, frequency, phase, and polarization
- Photodetector (PD): Produces current proportional to incident optical *power* $\boxed{I_{ph} = \mathcal{R} P_{opt}}$ $\mathcal{R}$=PD responsivity
- PDs are *insensitive to phase* of optical waves
- How to measure phase then? Using an interferometer

What is an interferometer?

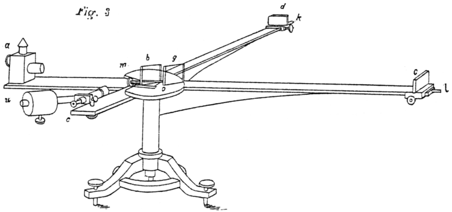
- Interferometers converts phase to intensity/power
- In GW detector context
  - optical phase difference $\propto$ differential strain: $\delta\phi = G\delta L$
  - converts $\delta\phi$ to intensity/power
  - Goal is to make G large

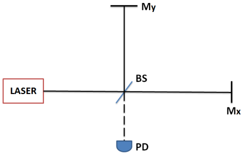


- ϕ_m =phase to be measured, ϕ_r =reference phase

Michelson interferometer layout

- Consists of light source, two arms with end mirrors, and beamsplitter





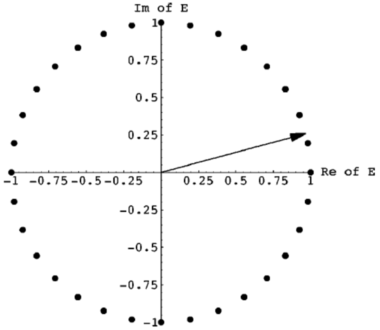
- Michelson interferometer from 1881; simplified optical layout

Maxwell's equations

- Classical light is electromagnetic phenomena; described by Maxwell's equations
  - Faraday's law: $\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial}{\partial t}\vec{B}(\vec{r},t)$
  - Ampere-Maxwell law: $\nabla \times \vec{H}(\vec{r},t) = \frac{\partial}{\partial t}\vec{D}(\vec{r},t) + \vec{J}(\vec{r},t)$
  - Gauss's laws: $\nabla \cdot \vec{D}(\vec{r},t) = \rho(\vec{r},t)$ and $\nabla \cdot \vec{B}(\vec{r},t) = 0$
  - Constitutive relations $\vec{D} = \epsilon\vec{E}$ and $\vec{B} = \mu\vec{H}$ encode medium properties
- Harmonic solutions: $\vec{E}(\vec{r},t) = \vec{E}_0 \cos(\omega t + \phi(\vec{r})) = \text{Re}[\vec{E}_0 e^{j\phi(\vec{r})} e^{j\omega t}]$
- $\underline{\vec{E}} = \vec{E}_0 e^{j\phi(\vec{r})}$ is called a **phasor**

Phasor representation

- Complex number, represented as a vector in complex plane
- Time-domain: $E\cos(\omega t+\phi) \to Ee^{j\phi} = \underline{E}$: Phasor
- Phasor: $\underline{E} \to \mathrm{Re}[\underline{E}e^{j\omega t}]$: Time-domain
- Exercise: Obtain phasor form of $\hat{x}\cos(\omega t-kz)+\hat{y}2\sin(\omega t-kz)$



E. D. Black and R. N. Gutenkust, AJP, 71(4), 2003

Describing optical waves: Plane wave description

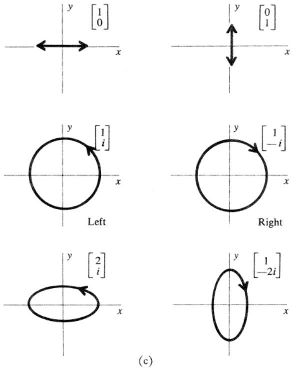
- From Maxwell's equations we obtain wave equation

$$\nabla^2 \underline{\vec{E}} \vec{r} + \omega^2 \mu\epsilon \underline{\vec{E}}(\vec{r}) = 0$$

- Optical waves propagating in $z-$direction; $\underline{\vec{E}}(\vec{r}) = \underline{\vec{E}}_T(x, y) A e^{-jkz}$
  - $k = \omega/c = 2\pi/\lambda$ is phase constant
  - $\underline{\vec{E}}_T(x, y)$ = transverse field distribution
  - **Plane wave**: $\underline{\vec{E}}_T(x, y)$ independent of x and y coordinates
  - Longitudinal part Ae^{-jkz} is a complex number at each z
  - $\underline{\vec{E}}_T$ determines **polarization** of wave
- Normalize such that $|A|^2$ is optical power

Polarization of light

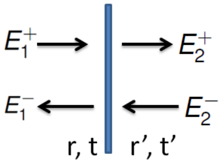
- Defined as orientation of electric field vector $\vec{E}$ in space
  - Linear polarization: $\vec{E}$ orientation constant with time
  - Elliptical polarization: $\vec{E}$ orientation varies with time
  - Jones vector: $\begin{pmatrix} A_x \\ A_y \end{pmatrix}$
- Optical elements such as quarter-wave and half-wave plates can be used to change polarization



G. R. Fowles, Introduction to Modern Optics

Describing mirrors

- Mirrors are used extensively in GW detectors and other optical systems
- Incident light partially reflected and transmitted by mirror
- Flexibility to choose phase of reflection and transmission coefficients; $\phi_r = \pi/2$ or π



Mirror matrix is unitary

$$M = \begin{pmatrix} jr & t \\ t & jr \end{pmatrix} \quad MM^\dagger = I$$

- $E_1^- = rE_1^+ + t'E_2^-$
- $E_2^+ = tE_1^+ + r'E_2^-$
- $|r| = |r'|$ and $|t| = |t'|$
- $r^*t' + t^*r' = 0$ and $|r|^2 + |t|^2 = 1$
- $R = |r|^2$ reflectivity and $T = |t|^2$ transmittivity

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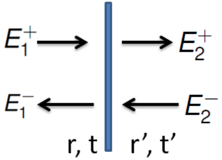

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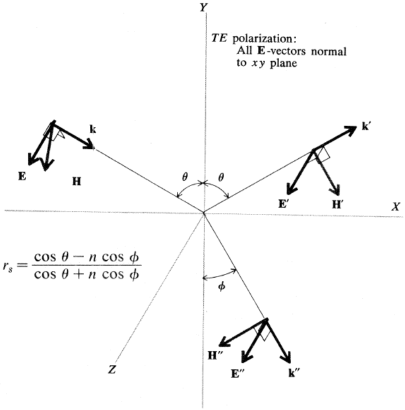
Mirror matrix is unitary

$$M = \begin{pmatrix} jr & t \\ t & jr \end{pmatrix} \quad MM^\dagger = I$$

Reflection and transmission coefficients

- Depend upon polarization of incident light, angle of incidence w.r.t. normal to interface, and refractive index on two sides of interface
- TE case: Electric field vectors are perpendicular to plane of incidence
- Coefficients can be derived by applying boundary conditions



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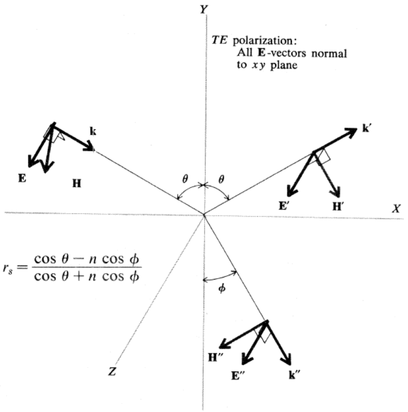
Boundary conditions

- At interface, vectors field vectors satisfy following conditions
  - Tangential $E-$field and normal $B-$field are continuous across boundary
  - Tangential $H-$field and normal $D-$field are discontinuous by amount of current and charge densities respectively
- In reflection coefficient calculation for TE case
  - $E + E' = E''$
  - $-H\cos(\theta) + H'\cos(\theta) = -H''\cos(\phi)$
  - $E/H = \eta/\sqrt{\epsilon}$



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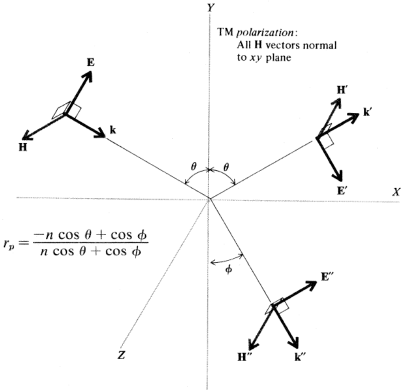
Reflection and transmission coefficients

- TM case: Magnetic field vectors are perpendicular to plane of incidence
- Coefficients can be derived by applying boundary conditions



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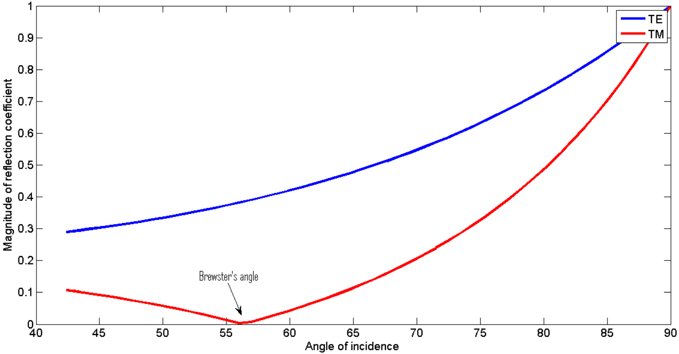
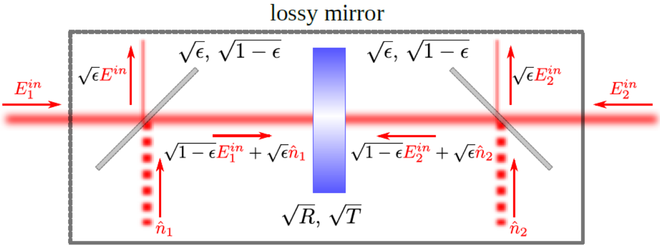
Reflection coefficients of TE and TM polarizations



- Zero reflection in TM case when light is incident at **Brewster's angle**
- This plot: $n_1 = 1$ (air) and $n_2 = 1.5$. What happens if $n_1 > n_2$?

Describing lossy mirrors

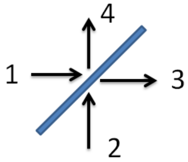
- Real mirrors are lossy due to absorption by mirror material
- Reflectance+Transmittance+loss=1, $|r|^2 + |t|^2 + L = 1$
- Further complication due to fluctuation-dissipation theorem which states that loss is accompanied by additional noise injected into system
- ϵ = absorption coefficient

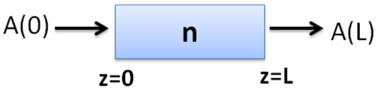


Danilishin and Khalili, LRR, 15 (2012)

Describing beamsplitter

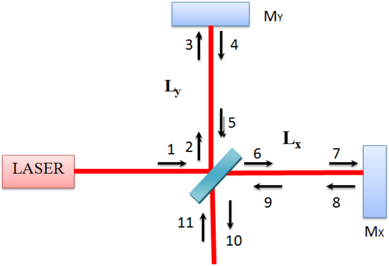
- I/O relation described by same matrix M
- Types: polarizing and non-polarizing
- Common 50:50 beamsplitter:
  $$B = \frac{1}{\sqrt{2}} \begin{pmatrix} j & 1 \\ 1 & j \end{pmatrix}$$
- **Delay**: $A(L) = A(0)e^{-jk_0 nL}$, accumulates phase delay $k_0 nL$ w.r.t $z = 0$





Layout and analysis of Michelson interferometer

- Beamsplitter splits laser light into two parts; one travels towards M_X other towards M_Y
- After reflection at mirrors $M_{x,y}$, beams recombine at beamsplitter
- $A_2 = \frac{j}{\sqrt{2}} A_1, \; A_5 = j r_Y e^{-j2kL_Y} A_2$
- $A_6 = \frac{1}{\sqrt{2}} A_1, \; A_9 = j r_X e^{-j2kL_X} A_6$



Interferometer output amplitudes

ASYM port: $A_{ASYM} = -\frac{1}{2}\left(r_X e^{-j2kL_X} - r_Y e^{-j2kL_Y}\right) A_1$

SYM port: $A_{SYM} = \frac{j}{2}\left(r_X e^{-j2kL_X} + r_Y e^{-j2kL_Y}\right) A_1$

Matrix analysis of Michelson interferometer

- Input vector at port 1: $\vec{\psi} = [A_1 \; 0]^T$
- Propagation+reflection+propagation towards beamsplitter described by matrix $P = \begin{pmatrix} jr_X e^{-j2kL_X} & 0 \\ 0 & jr_Y e^{-j2kL_Y} \end{pmatrix}$
- (Try) Multiply three matrices with input vector: $B^{-1}PB\vec{\psi}$ to get A_{ASYM} and A_{SYM}

Effect of gravitational wave

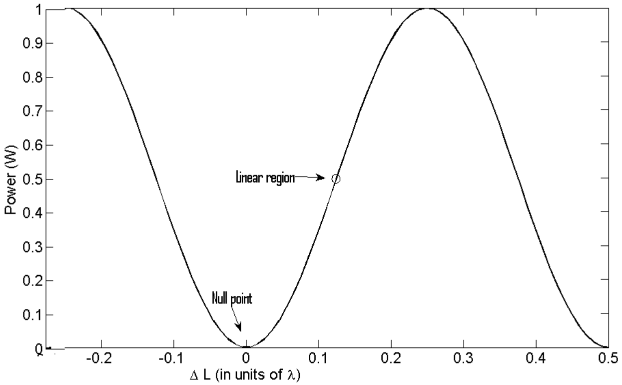
- GW perturbs mirrors and induces changes in reflected light
- $A_5 \to r_Y e^{-j2kL_Y} e^{-j2kL_Y h(t)/2} A_2$, $h(t)$ induces phase modulation
- Harmonic GW, $h(t) = h_0 \cos(\omega_{gw} t)$ creates sidebands

$$A_5 = A_2(0) \left(1 - \frac{jm}{2} e^{j\omega_{gw} t} - \frac{jm}{2} e^{-j\omega_{gw} t} \right)$$

- More about phase modulation later

Understanding interferometer response

- ASYM port amplitude: $A_{ASYM} = -\frac{1}{2}\left(r_X e^{-j2kL_X} - r_Y e^{-j2kL_Y}\right) A_1$
- Assume perfectly reflecting mirrors without loss: $r_X = r_Y = 1$
- ASYM port power $P_{ASYM} = P_{in}\sin^2(k\Delta L)$, $\Delta L = L_X - L_Y$



Operating in linear region

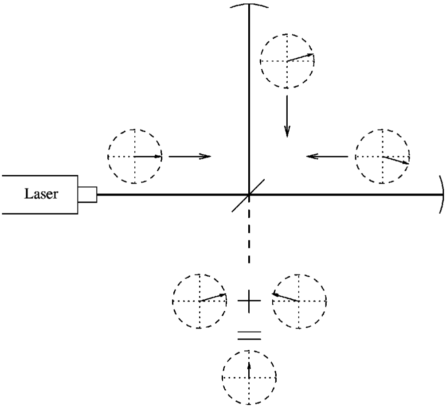
- Under GW perturbation, $L_X \rightarrow L_X + \delta l_X$ and $L_Y \rightarrow L_Y + \delta l_Y$
- Amplitude strain $h = \frac{\delta l_X - \delta l_Y}{L}$, $L = \frac{L_X + L_Y}{2}$ is avg. length
- $P_{ASYM} = P_{in} \sin^2(k\Delta L + khL)$. What should be $k\Delta L$ for operation in linear region?
- Expand P_{ASYM} using Taylor series with khL as perturbation

$$P_{ASYM} = P_{in} \sin^2(khL) + P_{in} khL \frac{\partial}{\partial(k\Delta L)} \sin^2(k\Delta L) + \cdots$$

- What value of $k\Delta L$ makes derivative maximum? (Ans. $\pi/4$)
- $P_{ASYM} \approx \frac{P_{in}}{2}(1 + 2khL)$; Laser intensity fluctuations swamps small signal (khL) term $\implies$ Linear region: bad!

Null region operation

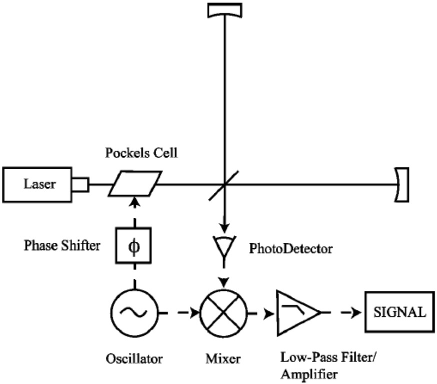
- At null point, $k\Delta L = 0$ so that $P_{ASYM} = P_{in}\sin^2(khL) \approx k^2h^2L^2$
- Since $h \ll 1$, $h^2 \ll 1$ makes detection a challenge
- Phasor analysis shows field exiting ASYM port is in quadrature ($\pi/2$) with respect to incident light
- Here beamsplitter and mirrors are assumed to provide 180° phase shift upon reflection



E. D. Black and R. N. Gutenkust, AJP, 71(4), 2003

Signal extraction using lock-in

- Modulate carrier to generate sidebands at λ_{mod}
- Make FP cavity dark only to carrier fields (Schnupp asymmetry)



Signal extraction using lock-in

$$k_{\pm} = \frac{\omega \pm \Omega}{c} = 2\pi\left(\frac{1}{\lambda} \pm \frac{1}{\lambda_{\mathrm{mod}}}\right)$$

$$t_{\pm} = i \sin\left[2\pi\left(\frac{\ell_x - \ell_y}{\lambda} \pm \frac{\ell_x - \ell_y}{\lambda_{\mathrm{mod}}}\right)\right] e^{ik_{\pm}(\ell_x + \ell_y)}$$

$$t_{\pm} = \mp i \sin\left[2\pi\left(\frac{\Delta\ell}{\lambda_{\mathrm{mod}}}\right)\right] e^{i[(\omega \pm \Omega)/c](\ell_x + \ell_y)}$$

$$P_{\mathrm{out}} = P_{\mathrm{in}} J_0^2(\beta) 4\pi^2 \frac{\ell^2}{\lambda} h^2 + 2 P_{\mathrm{in}} J_1^2(\beta) \sin^2\left(2\pi \frac{\Delta\ell}{\lambda_{\mathrm{mod}}}\right)$$

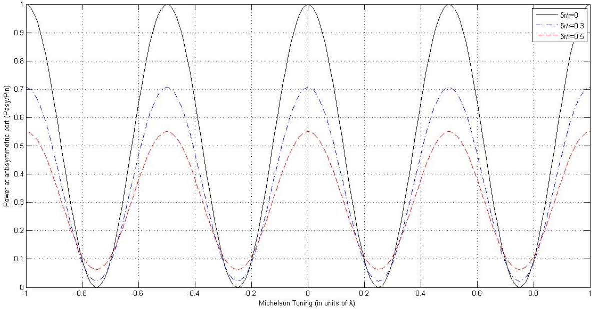
$$+ 2 P_{\mathrm{in}} J_1^2(\beta) \sin^2\left(2\pi \frac{\Delta\ell}{\lambda_{\mathrm{mod}}}\right) \cos\left(2\Omega t + 8\pi \frac{\ell}{\lambda_{\mathrm{mod}}}\right)$$

$$+ P_{\mathrm{in}} J_0(\beta) J_1(\beta) 4\pi \frac{\ell}{\lambda} h \sin\left(2\pi \frac{\Delta\ell}{\lambda_{\mathrm{mod}}}\right)$$

$$\times \cos\left(\Omega t + 4\pi \frac{\ell}{\lambda_{\mathrm{mod}}}\right).$$

Mirror reflection mismatch

- In practice $r_{X,Y} = r \pm \frac{\delta r}{2}$
- $P_{ASYM} = \frac{1}{4}\left[\left(r^2 - \frac{\delta r^2}{4r^2}\right)\cos^2(k\delta L) + \frac{\delta r^2}{4r^2}\right] P_{in}$



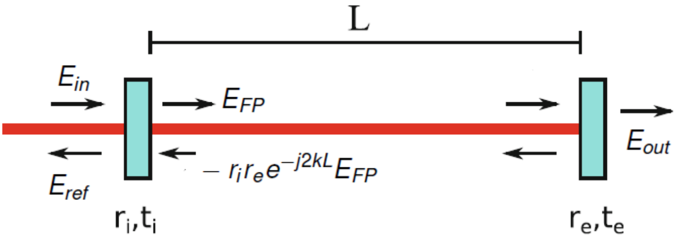
G. Vajente, Chap. 3, Advanced interferometers and the search for gravitational waves

Fabry-Perot cavity: layout

- Formed by two mirrors, $M_1 = \{r_i, t_i\}$ and $M_2 = \{r_e, t_e\}$, $t_e \approx 1$
- $-r_i r_e e^{-j2kL} E_{FP}$ fed back to cavity

$$E_{FP} = t_i E_{in} - r_i r_e e^{-j2kL} E_{FP} = \frac{t_i E_{in}}{1 + r_i r_e e^{-j2kL}}$$

- Output field: $E_{out} = t_e e^{-jKL} E_{FP} = \frac{t_i t_e E_{in}}{1+r_i r_e e^{-j2kL}}$



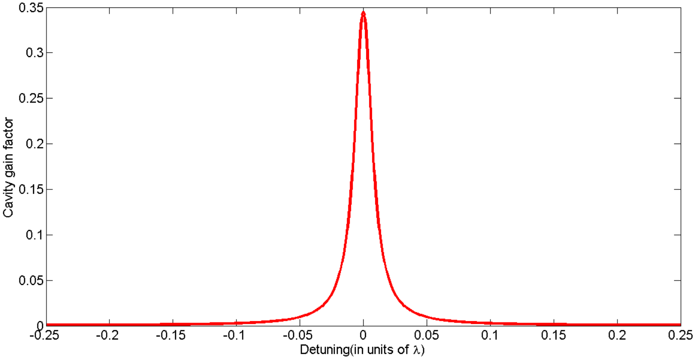
FP cavity: characterization

- At resonance, $e^{-j2kL} = -1$ and $P_{FP} = P_{in}\frac{(t_i t_e)^2}{(1-r_i r_e)^2} = G_{FP} P_{in}$
- Resonance condition implies multiple peaks spaced half-wavelength apart defining *free-spectral range FSR* $= \frac{c}{2L}$
- With detuning δL, $P_{FP} = \frac{t_i^2}{(1-r_i r_e)^2 + 4 r_i r_e \sin^2(k\delta L)} P_{in}$
- $\delta L_{FWHM} = \frac{\lambda}{2\mathcal{F}}$, where $\mathcal{F} = \frac{\pi\sqrt{r_i r_e}}{1-r_i r_e}$ is cavity *finesse*

$$P_{FP} = \frac{G_{FP}}{1 + \left(\frac{2\mathcal{F}}{\pi}\right)^2 \sin^2(k\delta L)} P_{in}$$

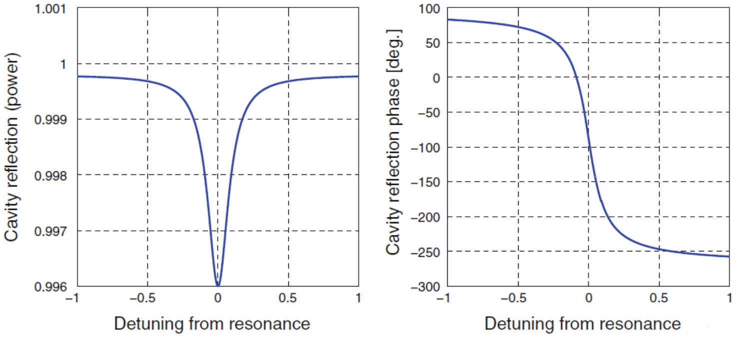
- Typical finesse values are >50

FP cavity: detuning



- In this plot, $\mathcal{F} = 30$ and $r_i = 0.9$. Calculate r_e and t_i

FP cavity: reflection



- $E_{ref} = j\frac{r_i + r_e(t_i^2 + r_i^2)e^{-j2kL}}{1 + r_i r_e e^{-j2kL}} E_{in}$
- In this plot, $\mathcal{F} = 30$ and $r_i = 0.9$. Calculate r_e and t_i

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FP cavity: reflection

- When $r_i = r_e$, at resonance, light is completely transmitted (**critical coupling**)
- When $r_i < r_e$, at resonance, light is reflected mostly but more importantly phase is highly sensitive to length variations (**over coupling**)
- To implement **power recycling**, FP cavities are operated in over coupling mode

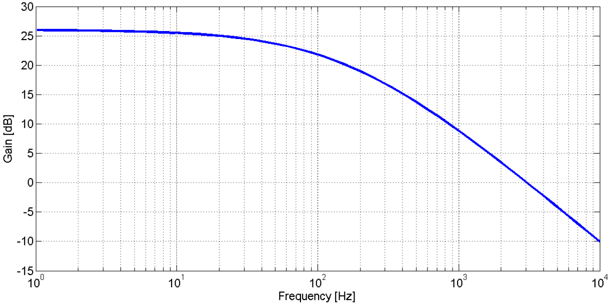
Mirror motion

- Mirror motion due to GW perturbation results in sidebands
- Displacement $x(t) = x_0 \cos(\Omega_s t)$ yields phase shift $\phi_s = 2kx(t)$
  $E_4 = jr_e e^{-j\phi_s} E_3 \approx jr_e E_3 + r_e \phi_s E_3 = jr_e E_3 + kr_e x_0 \left(e^{j\Omega_s t} + e^{-j\Omega_s t}\right) E_3$
- Sideband amplitude at resonance: $E_4(f_s) = \frac{kx_0 E_3(0)}{1 - r_i r_e e^{-j2(\Omega_s/c)L}}$

$$E_{ref} = \left[j\sqrt{G_{FP}} k r_e x_0 \frac{jt_i e^{-j(\Omega_s/c)L}}{1 - r_i r_e e^{-j2(\Omega_s/c)L}} \right] E_{in}$$

FP cavity: Frequency response

- For GW frequencies $\Omega_s L/c \ll 1$ so that $E_{ref} = -kr_e x_0 \frac{G_{FP}}{1+j\frac{f_s}{f_p}} E_{in}$, where $f_p = \frac{c}{4L\mathcal{F}}$ is critical frequency
- This is **low-pass filter** transfer function with low-frequency gain of $kr_e x_0 G_{FP}$ and 3-dB bandwidth f_p
- Since $f_p \propto \mathcal{F}^{-1}$, high finesse leads to lower bandwidth (why?)



Simulating FP cavities using Finesse

- Finesse is a frequency-domain simulation tool for interferometric detectors
- Easy to use and free!
- Latest version 2.0 released

Paraxial wave equation

- Practical optical beams are not plane waves; they are described by *paraxial wave equation*
- $(\nabla^2 + k^2)E(x, y, z) = 0$ with $E(x, y, z) = e^{jkz}A(x, y, z)$
- Paraxial approximation: $|\frac{\partial^2 A}{\partial z^2}| \ll \frac{2\pi}{\lambda} A$ gives equation $\left(\partial_x^2 + \partial_y^2 + 2jk\partial_z\right) A = 0$ describing propagation of beams

$$A(r, z) = \frac{1}{\sqrt{1+\frac{z^2}{z_R^2}}} e^{-\frac{x^2+y^2}{w^2(z)}} e^{-ik\frac{x^2+y^2}{2R(z)}} e^{i \arctan \frac{z}{z_R}} e^{-ikz}$$

$$w(z) = w_0\sqrt{1+\frac{z^2}{z_R^2}} \qquad R(z) = z\left(1+\frac{z_R^2}{z^2}\right)$$

$$z_R = \frac{kw_0^2}{2} \qquad \phi_G = -\arctan\frac{z}{z_R}$$

Gaussian beams

- Circularly symmetric with minimum transverse width w_0 at $z = 0$ known as beam waist
- $w(z)$ grows with z; at $z = z_R$, Rayleigh distance, $w(z) = \sqrt{2}w_0$

$$A(r, z) = \frac{1}{\sqrt{1 + \frac{z^2}{z_R^2}}} e^{-\frac{x^2+y^2}{w^2(z)}} e^{-ik\frac{x^2+y^2}{2R(z)}} e^{i \arctan \frac{z}{z_R}} e^{-ikz}$$

$$w(z) = w_0 \sqrt{1 + \frac{z^2}{z_R^2}} \qquad R(z) = z\left(1 + \frac{z_R^2}{z^2}\right)$$

$$z_R = \frac{k w_0^2}{2} \qquad \phi_G = -\arctan \frac{z}{z_R}$$

Higher-order modes

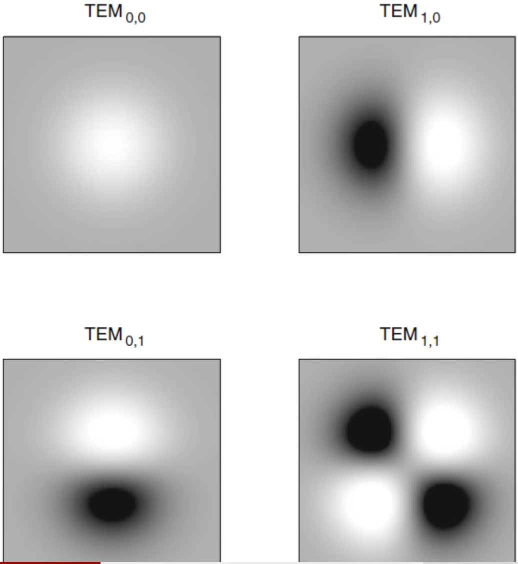
- Different solutions (modes) of paraxial equation; not necessarily cylindrical symmetric
- Common modes: Hermite-Gaussian or transverse electro-magnetic modes (TEM$_{mn}$)

$$\mathrm{TEM}_{mn}(x,y,z) = N_{mn}(z) e^{ikz} H_m\left(\frac{\sqrt{2}x}{w(z)}\right) H_n\left(\frac{\sqrt{2}y}{w(z)}\right) e^{-i(n+m+1)\arctan(z/z_R)} e^{ik\frac{x^2+y^2}{2R(z)}} e^{-\frac{x^2+y^2}{w^2(z)}}$$

$$N_{mn}(z) = \sqrt{\frac{2}{\pi w(z)^2\, 2^{n+m} m! n!}}$$

$$H_n(t) = e^{t^2}\left(-\frac{d}{dt}\right)^n e^{-t^2}$$

Higher-order modes



Resonators and beams

- Resonators cannot have plane surfaces (Why?); Stability of resonators depend on surface shapes
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